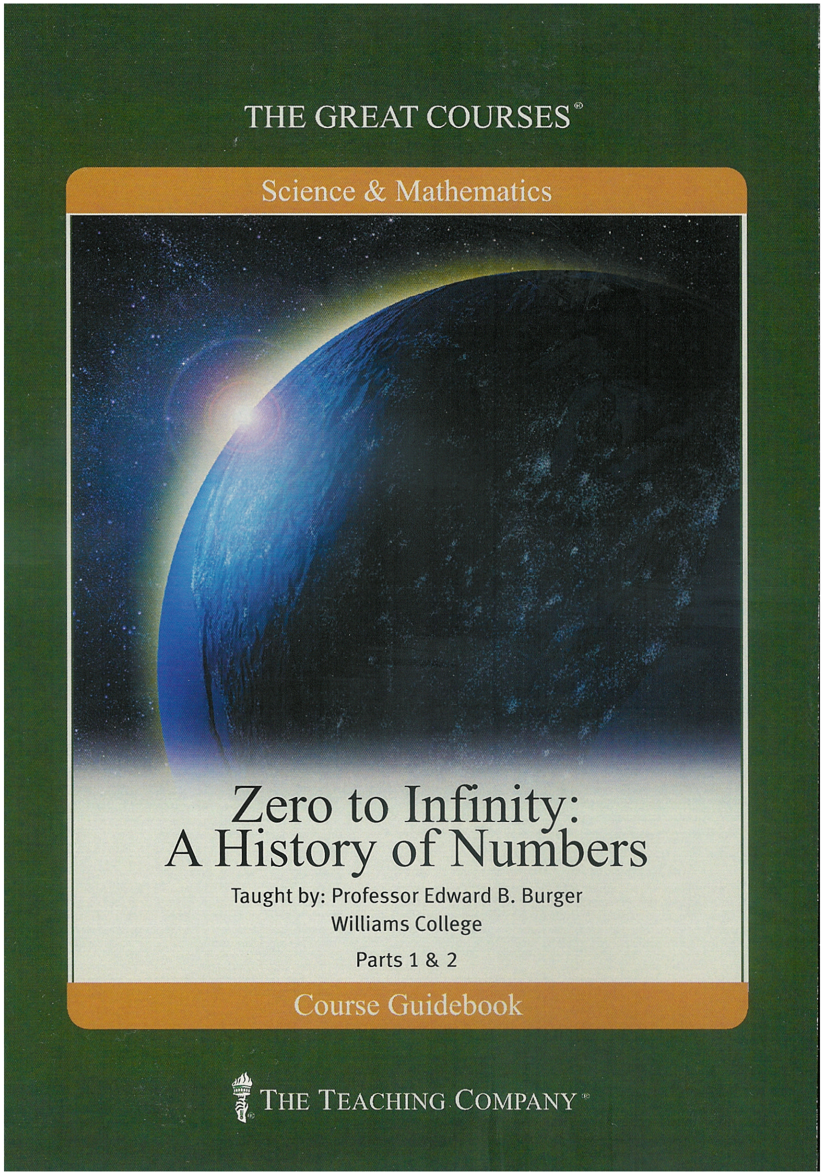

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Edward B. Burger is Professor of Mathematics in the Department of Mathematics and Statistics at Williams College. He graduated .*summa cum laude* from Connecticut College in 1985 earning a B.A. with distinction in Mathematics. He received his Ph.D. in Mathematics from The University of Texas at Austin in 1990. In 1990, he joined the faculty at Williams College. For the academic year 1990-1991, Dr. Burger was a postdoctoral fellow at the University of Waterloo in Canada. During three of his sabbaticals, he was the Stanislaw M. Ulam Visiting Professor of Mathematics at the University of Colorado at Boulder.

Professor Burger's teaching and scholarly works have been recognized with numerous prizes and awards. In 1987 he received the Le Fevere Teaching Award at The University of Texas at Austin. He received the Northeastern Section of the Mathematical Association of America Award for Distinguished Teaching of Mathematics in 2000 and in 2001 he received the Mathematical Association of America's Deborah and Franklin Tepper Haimo National Award for Distinguished College or University Teaching of Mathematics. In 2003, he received the Residence Life Academic Teaching Award at the University of Colorado. Professor Burger was named the 2001-2003 George Polya Lecturer by the Mathematical Association of America. In 2004 he was awarded the Chauvenet Prize—the oldest and most prestigious prize awarded by the Mathematical Association of America. In 2006, the Mathematical Association of America presented him with the Lester R. Ford Prize. In 2007, Williams College awarded him the Nelson Bushnell Prize for Scholarship and Teaching: that same year, he received the Distinguished Achievement Award for Educational Video Technology from the Association of Educational Publishers. In 2006 Professor Burger was listed in the *Reader's Digest* annual "100 Best of America- special issue as "Best Math Teacher."

Professor Burger's research interests are in number theory, and he is the author of 12 books and more than 30 papers appearing in scholarly journals. With Michael Starbird, he coauthored *The Heart of Mathematics: An Invitation to Effective Thinking*, which won a Robert W. Hamilton Book Award in 2001. They also coauthored a general-audience trade hook titled *Coincidences, Chaos, and All That Math Jazz*.

In addition, Professor Burger has written seven virtual video CD-ROM textbooks with Thinkwell and has starred in a series of nearly 2.000 videos

that accompany the middle school and high school mathematics programs published by Holt, Rinehart and Winston He has served as chair of various national program committees for the Mathematical Association of America and as associate editor of the American Mathematical Monthly. He is also a member of the board of trustees of the Educational Advancement Foundation.

Professor Burger is a renowned speaker and has given more than 400 lectures around the world. His lectures include keynote addresses at international mathematical conferences in Canada, France, Hungary, Japan, and the United States: mathematical colloquia and seminars at colleges and universities; presentations at primary and secondary schools; entertaining performances for general audiences; and television and radio appearances on WABC-TV, the Discovery Channel, and National Public Radio.

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Zero to Infinity: A History of Numbers

Zero to Infinity:
A History of Numbers

Scope:

In this course, we will paint a picture of an ever-evolving notion of number—coming to the realization that the notion of number is a difficult if not impossible idea to define precisely. Our course will have two main points of focus: the historical evolution of the representation of numbers for communication and manipulation—"the numbers in life"—and the intrinsic structure of those numbers—"the life of numbers." We will discover throughout the course that these two perspectives synergistically inform each other and allow our understanding to grow and evolve with the numbers themselves. Although we will explore these two vantage points and their interconnectedness, we will also offer the recurring historical theme of number as a means to count, quantify, measure, and compare—and surprisingly, all of our intuitions about these basic notions will be challenged.

The Numbers within Our Lives (four lectures)

Starting with the fundamental desire to better understand our world by quantifying it, we will study some of the early attempts to determine "how many?" We will explore clever methods that were used before any abstract notion of number existed. Curiously, as we will discover at the end of this course, these early non-numerical methods hold the keys to unlocking infinity—a realization that set the mathematical world ablaze.

We will then explore the dawn of numerate humanity and celebrate the intellectual triumph of moving number from an adjective ("three" apples) to an abstract noun ("three"). This shift led to a desire to express these new abstract objects in useful and meaningful ways: thus, we will also study the evolution of humankind's struggle with representing and naming numbers. It is here that we will see the dawn of zero, a surprisingly subtle and important concept. We will close this part of the course with a discussion of how numbers have captured the imagination of curious individuals throughout the ages. The allure of numbers has led not only to recreation but also to the beliefs that numbers have personalities and possess spiritual and magical powers.

The Lives within Our Numbers (six lectures)

We will open this part of the course with the discovery that number patterns existed long before our desire to represent them—thus, we see that numbers and the patterns they hold transcend our imagination and are a foundational part of nature and our universe. Once we name the abstract objects known as numbers, however, they are born, take on a life of their own, and attract the interest and capture the imagination of curious minds throughout human history. In this part of our course, we will explore that historical journey to discover and appreciate the structure and beauty of numbers in their own right

We will look at famous collections of numbers, such as the Fibonacci numbers and prime numbers. We will then move into the fractional world of rational numbers and the much more mysterious and confounding universe of irrational numbers. Even though the ancient Greeks proved that these objects exist, they were unable to view them as numbers. Armed with both rational and irrational numbers, we will explore how these two different forms fit together within the real number line. Finally, we will consider another famous collection of real numbers that exhibits some extremely beautiful albeit counterintuitive properties.

Transcendental Meditation—The π and e Stories (four lectures)

We will open this part of the course by briefly reviewing the historical underpinnings of arithmetic and algebra and their influence on the theory of numbers. We will then share the wonderful stories of the celebrated numbers π and e, discovering not only the histories of these important constants but also their significance in mathematics and beyond.

With these two famous numbers as our inspiration, we will walk the intellectual path of mathematicians toward a better understanding of such numbers as π and e. This exploration will lead us to the subtle concept of transcendental numbers. Here, we will see an illustration of an important recurring phenomenon: That which first appears to be bizarre and strange is, in actuality, normal and commonplace.

Algebraic and Analytic Evolutions of Number (four lectures)

With our initial understanding of numbers in place, we will offer two elegant mathematical perspectives on how numbers have evolved. The algebraic point of view will lead us naturally to "imaginary" numbers—which will then appear far less imaginary. Armed with the imaginary number i, we will discover one of the most amazing and beautiful formulas in mathematics—one that connects the five most important numbers together into one incredible equality.

The analytical evolution of number—based on measuring distances and closeness—will lead us to new numbers that defy our intuition of what - number" should mean or how numbers should look and behave. Paradoxically, these strange and foreign modern numbers have provided physicists with new insights into quantum physics and our not-so-foreign universe.

Infinity—"Numbers" beyond Numbers (five lectures)

In this final part of the course, we will travel beyond numbers and contemplate the enormous question. what comes after we have exhausted all numbers? We will tame infinity and discover that, contrary to our initial intuition, infinity shares some basic properties with number—in particular, they both come in different sizes.

The wonderful and surprising realization in these lectures will be that the dramatic and at-first highly controversial theory of infinity arises from the same principle humankind first employed to "count.- Thus, we will come full circle, discovering that although the ideas of infinity carry us to levels of abstraction and imagination that most individuals dare not reach even today, the nucleus of that incredible theory relies on an insight of our ancestors from 30,000 years ago. This journey to infinity and beyond will be the final brushstroke on our painting of the endless frontier of the notion of number.

Lecture One
The Ever-Evolving Notion of Number

Scope: In this lecture, we will introduce the concept of number and foreshadow an interesting paradox: Although numbers are precision personified. a precise definition of number still eludes us. In fact, one of the central themes of this course is that the concept of number is not a fixed, rigid idea but an ever-evolving notion. As our understanding of the world expands and our capacity for abstract thinking grows throughout history, so too does our view of what number means. We will see numbers move from useful tools for measuring quantities to abstract objects of independent interest. This lecture previews the main themes of the course, from an exploration of the life of numbers to the endless world of the transfinite.

Outline

I. Welcome to a world of number.

   A. What is your definition of number?

   B. The distinction between number and numbers is subtle.

   C. Numbers are at once practical notions in our everyday world and abstract objects from our imagination.

   D. Before our ancestors could write, they contemplated quantities.

   E. Historically, the study of numbers was a central component of one's education—one of the original liberal arts.

II. Many people incorrectly believe that mathematics is completely understood; most of mathematics, in fact, remains mysterious.

   A. Forward progress is extremely slow moving.

   B. New discoveries in mathematics are made by building on the work of others who came before.

III. Our knowledge of the early origins of number is vague; we must depend on relics that archaeologists uncover.

   A. Some ancient civilizations recorded their work on materials that stood the test of time.

   B. Others employed materials that, over time. disintegrated; thus, our knowledge is as fragmented as the ancient. broken tablets we try to

understand.

C. In this course, we will study moments in time to produce a mosaic of small pieces that. when viewed from afar, will allow us to see how numbers grew in our understanding and sparked our imagination.

IV. This course is a blend of mathematics in a historical framework.

A. Although these lectures offer a fluid conceptual development of the notion of number. at times we will gently glide back and forth through history so that we can appreciate and better understand the allure of number as our story unfolds.

B. The course covers three main themes.

C. We will discover that numbers are truly difficult to define precisely, despite what most people believe.

1. We will come to appreciate the notion of number as one that is always evolving.

2. We will also see the recurring theme that what at first appears familiar and commonplace is, in fact, rare and exotic; conversely, what first appeared exotic will later be viewed as the norm.

D. The first series of lectures focuses on early attempts to quantify

1. We will journey back to 30.000 B.C.E. and see some of the earliest attempts to count.

2. Numbers slowly evolved into adjectives (e.g., "three" apples).

3. Counting numbers (also known as natural numbers) became the most familiar numbers (e.g., 1, 2, 3, 4, 5 ...).

E. We will investigate the challenges of communicating and manipulating numbers.

1. Through these investigations, we will see the notion of number expand further, as our ancient ancestors struggled with zero and negative numbers.

2. We will explore how individuals were moved to associate personalities, magic, and even cosmic significance to numerical notions.

F. We will explore numbers in nature and discover how Fibonacci strove to make them more natural. We will then focus on the nature

of numbers themselves.

1. By the 6th century B.C.E., the Pythagoreans were studying numbers as objects in their own right, rather than using them , solely as tools for calculation and recordkeeping.

2. Pythagoras may have been inspired by the religious sect in India known as the Jains, whose members may have been the first number theorists. Today, this exploration into the study of numbers is known as number theory.

3. Cultures share their passion for number theory; the more we explore, the more our field of vision of number widens.

G. We will celebrate two of the most important numbers in our universe, π and e, using these famous quantities as the inspiration to see subtle distinctions between different types of numbers.

H. We will consider two mathematical views of number evolution that allow us to expand our notion of number in new directions. We will also encounter "numbers" that challenge our very notion of what number means.

I. We will journey beyond the universe of number and delve into the more abstract world of infinity. Using the very first method for counting. we will discover that, just as with numbers. infinity can be understood and can hold many surprising features.

1. Although our discussions will become a bit technical at some points, those details are not the central focus of this course.

2. Our main goal is the realization that the study of number is a beautiful endeavor that has captured humankind's imagination throughout the ages and continues to inspire us to explore its endless frontier.

Questions to Consider:

1. What is your definition of number? You are encouraged to write down your definition after this lecture to see how it changes throughout the course.

2. For what purpose are numbers used? What types of numbers have you encountered in your life?

Lecture Two
The Dawn of Numbers

Scope: Humans have an innate capacity to accurately compare small quantities. We will examine some early counting tools as a means to determine how humanity's understanding of numbers initially developed. Humankind has been counting for at least 30.000 years, but are humans the only creatures to possess a number sense? In this lecture. we will see that even some animals appear to have the capacity for numerical concepts. The human concept of number may have developed in the same way it does in children. To compare large quantities, however, early civilizations used the idea of a one-to-one pairing. Notched bones, knotted strings, and piles of pebbles allowed people to keep track of animals and conduct commerce. Although the human hand is one of the most fundamental counting tools studies of primitive cultures reveal the subtle use of the entire body in counting practices. Next, we will turn to the development of the abstract notion of number; when, for example, did the adjective three (e.g., "three" apples) become the noun three? Although this event did not occur at a precise moment, evidence of abstract numbers in Mesopotamia dates back somewhere between 3500 and 3200 B.C.E. (dates range considerably).

Outline

I. What motivated humans to count?

A. Thousands of years before there were writing, literacy, or even numeral symbols, shepherds tending flocks had to keep track of their sheep.

B. As agricultural societies developed, people needed to measure and divide land, keep track of livestock, record harvests, and take census data.

C. With growing populations and clashing cultures came conflict, requiring armies to face the logistics of arming and feeding their soldiers.

D. Bountiful agricultural fruits of labor required counting days and lunar cycles as part of calendars to better predict the change in seasons, annual floods, or dry spells.

II. Human beings have an innate number sense.

A. This innate number sense allows us to instantly compare small collections of objects.

1. If a Sumerian shepherd has a very small number of animals, he can keep track of them without the need for counting.

2. It is easy to see the difference between a herd of four sheep and a herd of three without actually counting.

3. With a larger herd, we are unable to determine (by simply looking) whether the collection of sheep we have after grazing is the same size as the collection with which we started. This limitation is referred to as the limit of four.

4. This limit of four might underlie the barred-gate system of counting we still employ today.

B. Other creatures also sense numbers.

1. Studies with goldfinches reveal that when presented with two small piles of seeds, they usually pick the larger of the two piles; crows have also been known to distinguish between collections of different sizes.

2. Evidence suggests that animals do not, however, have a notion of number as an abstract object.

C. The human concept of number may have developed in a manner similar to the way in which it develops in children.

1. Ordination comes first; that is, the ability to see that one set of objects is larger than another. We learn to order objects according to size before we learn to count them.

2. Learning ordered lists is a classic component of early education; children are taught to recite the alphabet, numbers, and even the days of the week, often well before they understand the meaning of these sequences.

3. Children next begin to grasp the idea of natural numbers (e.g., 1, 2, 3, 4 ...).

4. Finally, children master cardination (or true counting), in which the objects in one collection can be counted or paired up with objects from another collection.

III. Many societies used various forms of sticks as counting tools to record one-to-one correspondences.

A. Notched bones from as long ago as 30.000 B.C.E. have been found

in Western Europe.

B. Notched sticks called tally sticks have been used for millennia and may have inspired the development of Roman numerals. A wooden tally stick could be marked and then split lengthwise so that two parties could keep track of a transaction.

C. In order to make the one-to-one correspondence physical, the Incas and cultures along the Pacific Rim and in Africa used knotted strings.

1. In the 5th century B.C.E., Herodotus of Greece wrote in his History that Darius, the king of Persia, used a knotted cord as a calendar.

2. Catholic, Muslim, and Buddhist rosaries and prayer beads allow the devout to recite the appropriate number of litanies without the need for an abstract counting system.

D. As early as 3500 to 3200 B.C.E.. our Sumerian shepherd most likely used a pile of pebbles to "count" his sheep through a one-to-one pairing.

IV. The human hand is a natural counting tool.

A. The limit of four in humans made the five-digit hand particularly useful for counting and led to the use of five as a basic grouping for counting.

1. The continued popularity of the barred-gate tally system may be due to its basis in counting by 5s.

2. The 10 digits on our two hands may have led to the modern-day dominance of a base-10 numeral system.

B. Toes are the obvious extension of the hand as a counting tool.

Given the hand's convenience, some cultures extended their counting to include the joints of fingers.

Other body parts have been incorporated into counting systems in many cultures.

V. Sumerian methods of counting have been studied extensively.

A. Sumerians created different clay tokens, called calculi. to represent quantities of different items.

1. To record a quantity of goods, such as measures of grain in a storehouse, Sumerians would seal the appropriate collection of

tokens in a clay jar. Evidence for this method of counting can be found as early as 3200 B.C.E.

2. The tokens were used to make impressions on the outside of the jar before the clay hardened, indicating the quantity within.

3. The token markings later came to represent the numbers themselves, eliminating the need for the tokens in recordkeeping.

B. Sumerian markings led to one of the first numeral systems and, consequently, to what may have been the first form of written language: cuneiform.

Questions to Consider:

1. Without counting, determine whether the collection of @ signs below or the collection of & signs is larger.

@@@@@@@@@@@

& & & & & & & & & & & & & & &

How were you able to perform this task without counting?

Now determine without counting whether the collection of @ signs below or the collection of backslashes is larger.

@@@@@@@@@@@

\\\\\\\\\\\\\\\\\\\\

Discuss why one comparison was easy and the other more difficult.

2. Find examples in your everyday life where you use a one-to-one pairing to compare the sizes of two collections without actually counting.

Lecture Three
Speaking the Language of Numbers

Scope: In this lecture, we will examine the words and symbols that early cultures employed to represent numbers as abstract quantities. Many advanced societies, such as the Sumerians Egyptians, Chinese, and Mayans, developed sophisticated numeral systems. Because numbers arose out of utility, some naming conventions are limited. Number theory was born in India but developed in Greece near the same time Negative numbers first appeared in the 1st century B.C.E Even though early societies artfully employed numbers to create sophisticated calendars, keep track of agricultural production, and record commercial transactions, they still struggled with complex arithmetic. The desire to calculate led to the development of arithmetic aids that may have laid the groundwork for the place-based number system we use today. The age-old struggle to perform complex arithmetic ran parallel with the age-old struggle to devise an appropriate means to represent numbers.

Outline

I. Number systems evolved at different times throughout the world.

A. Around 3500 B.C.E., Sumerian round jars evolved into flat tablets with symbols known as pictographs, which might be the moment when numbers were first viewed as nouns rather than simply as adjectives.

1. These pictographs created by using a reed-stem stylus to mark in clay, were developed by the Sumerians and Babylonians (collectively known as Mesopotamians).

2. The reed-stem stylus gave way to a pointed stylus for creating more delicate shapes.

3. This style of creating number symbols evolved into cuneiform and may be the first instance of writing in human history.

B. The Babylonians began with a system that used symbols for powers of 60 (1, 60, 3600 ...), 60 being the smallest number that can be divided evenly by 2, 3, 4, 5, and 6 During the 2nd millennium B.C.E., the Babylonians developed a positional system; by 300 B.C.E., they had a symbol for zero as a placeholder.

C. The Egyptians also employed pictographs for numbers as early as 3500 B.C.E.

1. With the invention of papyrus, the Egyptians were able to record more complex bookkeeping methods and calculations.

2. The famous Rhind Papyrus (c 1650 B.C.E.) contains many calculations and includes hieroglyphs for addition and subtraction.

D. The Chinese used a system involving powers of 10 as early as 1400 B.C.E., with symbols for 1 through 10, in addition to 100 an 1000.

1. Few records survived the humid Chinese climate, making the culture's history of number development and use more uncertain.

2. The ancient Chinese used the words for large numbers in a poetic manner to create new words.

E. E The Mayans of Central America had a numeral system by the 3rd century C.E.

1. Their system involved powers of 20 and used symbols that consisted of dots and lines Dots and dashes still remain with us today—consider the symbol for division ÷.

2. The Mayan system, though compact, was still cumbersome for calculation.

3. The Mayans had a symbol for zero but used it solely as a placeholder.

II. Numbers arose out of utility; thus, some naming conventions are limited.

A. Primitive cultures may function very well with words only for one two, and many According to 19th-century anthropologists, the Botocudos from Brazil would count up to four and express many by pointing to their hair, implying that, beyond four, things were as countless as the hairs on one's head.

B. For the Lengua people of Paraguay, words for numbers reflect body counting.

1. The word for ten literally means "finished both hands."

2. The word for fifteen means "finished foot."

C. In Indonesia, lima means “five” even though it literally means “hand.”

D. In some languages, the word for *whole person* also means "twenty."

E. The Spanish word ties ("three") and the French word trés ("very much") derive from the Latin word tres, which means, among other things, "beyond." The word three in English is derived from the Anglo-Saxon thria, which is related to the word throp, meaning "pile" Or -heap."

F. In the days of Romulus (c 750 B.C.E.), the Roman calendar had 10 months, starting with March.. Months 5 through 10 were named Quintilis Sextilis, September, October, November, and December, designating their position in the year January and February were added later and Quintilis and Sextilis were renamed Julius and Augustus in honor of two Caesars.

III. Number theory was born in the 4th century B.C.E.

A. At around this time, members of a religious community in India called the Jains were exploring mathematics with a new, sophisticated perspective.

1. The Jains classified numbers into different categories, including at least four different types of infinity.

2. In some sense, the Jains were the first number theorists—that is, the first group of individuals to study number as an independent, abstract object of interest.

B. Pythagoras explored the abstract notion of number in great depth in the 6th century B.C.E.

1. Pythagoras might have been inspired by the work of the Jains.

2. The Pythagoreans were among the first to consider numbers as abstract objects, rather than tools, and the first to prove theorems involving numbers.

C. This period truly represents the birth of the life of numbers

IV. When did negative numbers enter the scene?

A. In the 1st century B.C.E. the Chinese solved systems of equations simultaneously using negative numbers.

B. There is evidence of calculation with negative numbers in India between 200 B.C.E and 200 C.E.

C. In the 3rd century C.E., the Greek mathematician Diophantus described an equation equivalent to 4x / 20 = 0 as "absurd" because its solution is a negative number (x = –5).

D. In 7th-century India, negative numbers were used to represent debts The great Indian astronomer Brahmagupta (598–c 665 C.E.) was the first to offer a systematic treatment of negative numbers.

E. As late as the 18th century, some important mathematicians discarded negative solutions because they viewed them as unrealistic Today, we define integers as the collection of all natural numbers, their negatives, and zero (e.g., . –3, –2, –1, 0, 2, 3 ...).

V. Illustrations of ancient arithmetic abound.

A. One of the oldest recorded division calculations dates from 2650 B.C.E in Sumeria.

1. The challenge was: A granary has a given amount of barley to be distributed so that each man gets 7 *sila* (a measurement unit) of barley Find the number of men.

2. Historians speculate that this calculation was performed using calculi.

B. To multiply, the Egyptians employed a technique called doubling They also performed calculations involving fractions with numerator 1 (unit fractions) This convention limited their understanding of more complicated numbers.

C. The Babylonians computed square roots as early as 1800 B.C.E.

1. They were able to produce a very good approximation to $\sqrt{2}$

2. They understood the Pythagorean Theorem by 1600 B.C.E. more than 1000 years before Pythagoras.

VI. What is a calculator?

A. Before the invention of modern calculation devices, the word "calculator" referred to an individual who performed calculations.

B. B Many ancient cultures performed calculations on a sand table.

1. Pebbles and other markers were used in columns representing different numbers that were then moved to depict addition or subtraction.

2. Another possible origin for the zero symbol (0) might be the round dimple left in the sand when a pebble was removed, leaving an empty column.

C. The classic Chinese abacus came into use in China during the 14th century C.E.

1. Long before that time, as early as 450 B.C.E., the Chinese as well as the Egyptians, Romans Greeks and Indians, used a simpler calculating board, similar to sand tables.

2. The ancient Greek historian Herodotus (c 484–c 425 B.C.E.) is credited with the observation that "the Egyptians move their hands from right to left in calculations while the Greeks from left to right."

3. The abacus allows for fast calculation in an additive numeral system and is still in use today.

VII. Modern Hindu-Arabic numerals arose from a mysterious mixing of traditions from two cultures.

A. The origin of numerals can be traced back to India and evolved from 400 B.C.E to 400 C.E.

B. They grew out of Brahmi numerals.

C. These symbols were introduced to European nations through the writing of Arab and Persian mathematicians and scientists.

D. Although we have a clear understanding of the origins of some numerical symbols (e.g., 1, 2, 3) historians disagree on the origins of other numerals.

E. Writing these numerals became a form of art as seen in the work of Albrecht Dürer.

F. The 10 Hindu-Arabic numerals form an efficient basis for our current decimal system.

Questions to Consider:

1. Suppose you have an additive numeral system in which 1 5 and 25 are denoted by |, @, and %, respectively What numbers do the following represent?

   %%@@@||, %@||||, %%%, @@@@

2. Create a context in which negative numbers would be absurd.

Lecture Four
The Dramatic Digits—The Power of Zero

Scope A compact place-based (positional) numeral system with a symbol for zero opens the floodgates for arithmetic calculations and the discovery of new numbers. Here, we will explore the origins of zero and the development of our modern decimal system. With powerful positional numeral systems in place, humankind finally had the tools necessary to begin the development of modern mathematics. Along with the commonly used decimal system, we will consider the binary number system, a system with roots in ancient China and India that is the basis for the modern computer. We contemplate other base systems, and we close with a look at the echoes of ancient positional systems in our modern ones.

Outline

I. Ancient numeral systems were mostly additive.

A. Most of the systems we studied in the previous lecture required the repetition of symbols (e.g., XXIII for 23 in Roman numerals).

B. Although computation with additive systems was fast, thanks to such tools as the abacus, those systems required very long lists of symbols to denote larger numbers.

C. Additive systems made it difficult to look at more arithmetically complicated questions, thus slowing the progress of the study of numbers.

II. For millennia, zero did not count as a number.

A. In the Rhind Papyrus from 1650 B.C.E., numbers were referred to as "heaps."

1. This tradition continued with the Pythagoreans, who in the 6th century B.C.E. viewed numbers as "a combination or heaping; of units."

2. Aristotle defined number as an accumulation or "heap."

3. Because we cannot have a "heap" of zero objects, zero was not viewed as a number.

B. The lack of zero resulted in many challenges.

1. A careless Sumerian scribe could cause ambiguities: In cuneiform, different spacing between symbols led to different numbers.

2. The Egyptian system did not require a placeholder such as zero, but the Egyptians' additive notation was cumbersome: in 2000 years, little progress was made in arithmetic or mathematics.

C. Though the Babylonians and the Mayans had symbols for zero, they considered it a placeholder rather than a number.

D. As noted earlier, the modern symbol 0 may have arisen from the use of sand tables for calculation.

1. Calculations performed on sand tables may have led to a broader use of place-based number systems.

2. Ptolemy later used the Greek letter o (omicron) to denote "nothing," although he did not view it as a number.

E. In 7^{th}-century India, the astronomer Brahmagupta understood zero as a number, not simply as something left from removing a counter in a sand table, he studied $\frac{0}{0}$ and $\frac{1}{0}$ deciding (erroneously) that $\frac{0}{0}$ = 0, he did not know what to conclude about $\frac{1}{0}$.

III. Positional systems can use different bases.

A. The base-10 or decimal system uses the l0 Hindu-Arabic numerals: 0. 1. 2. 3. 4. 5. 6. 7. 8 and 9 (called digits).

1. The position of a digit indicates by which power of l0 that digit is to be multiplied.

2. We will use exponential notation to denote powers of l0: l0 x10 $=10^2$. l0 x l0 x l0 $=10^3$, and so forth; l0° = 1.

3. For example:

$293 = (2 \times 100) + (9 \times 10) + (3 \times 1)$

$= (2 \times 10^2) + (9 \times 10^1) + (3 \times 10°)$

$8105 = (8 \times 1000) + (1 \times 100) + (0 \times 10) + (5 \times 1) = (8 \times 10^3) + (1 \times 10^2) + (0 \times 10') + (5 \times 10°)$.

4. A specific mark, called a radix (or decimal) point, is used to separate the whole-number positions from the fractional positions.

5. In some countries (including the United States), this mark is period:

$$3.14 = (3 \times 1) + (1 \times \frac{1}{10}) + (4 \times \frac{1}{100}) =$$

$$(3 \times 10°) + (1 \times \frac{1}{10}^1) + (4 \times \frac{1}{10}^2)$$

6. In some other countries, the separator is a comma:

$$87{,}05 = (8 \times 10^1) + (7 \times 10°) + (0 \times \frac{1}{10}) + (5 \times \frac{1}{100})$$

B. The base-2 (or binary) system has only two digits: 0 and 1.

1. In this system, the positions are valued as powers of 2 rather than powers of 10.

2. Consider this example:

$1011_2 = (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1)$

$= (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2°)$.

The number 1011_2 equals 8 + 2 + 1 = 11 in base l0. The subscript 2 denotes that the number is expressed in base 2.

3. Fractional positions are analogous to those in base 10. For example:

$$101.11_2. = (1 \times 4) + (0 \times 2) + (1 \times 1) + (1 \times \frac{1}{2}) + (1 \times \frac{1}{4})$$

$$= 1 \times 2^2) + (0 \times 2^1) + (1 \times 2°) + (1 \times \frac{1}{2}^1) + (1 \times \frac{1}{2}^2).$$

The base-2 number 101.11_2 equals $4 + 1 + \frac{1}{2} + \frac{1}{4}$ in base 10.

C. The base-3 (or ternary) system has only three digits: 0. 1. and 2.

1. The positions are valued as powers of 3.

2. Consider this example:

3. $2102_3 = (2 \text{ x } 3^3) + (1 \text{ x } 3_2) + (0 \text{ x } 3^1) + (2 \text{ x } 3^{\circ})$.

4. The number 2102; equals 54 + 9 + 2 = 65 in base 10.

5. Fractional positions are once again analogous to those in base 10. Consider:

$$12.202_3 = (1 \text{ x } 3^1)+(2 \text{ x } 3^{\circ})+(2 \text{ x } \frac{1}{3})+(0 \text{ x } \frac{1}{9})+(2 \text{ x } \frac{1}{27}),$$

The base-3 number 12.202_3 equals $3+ 2 + \frac{2}{9} + \frac{2}{27} = 5 \frac{20}{27}$ base 10.

D. Any whole number greater than 1 can be used as a base; given any base, any number can be expressed in that base.

1. The Babylonians used base 60.

2. Some computer languages use base 16. called hexadecimal.

E. From the 13th through the 16th centuries, Europeans disagreed about the advantages of a positional system versus an additive system.

1. An additive system allowed speedy calculation using an abacus and did not require memorizing multiplication tables.

2. Some worried that a positional system was more vulnerable to fraud because someone could radically change the value of a number by adding a single digit at one end or the other.

3. The dominance of the positional system has proved enormously valuable, not just in the efficient writing of numbers but in the advancement of the theory of numbers and mathematics in general.

IV. We find harbingers in ancient times of some modern positional systems, along with holdovers of some very old practices in our lives today.

A. The Chinese I Ching from 2800 B.C.E. contains patterns of solid and broken lines (called trigrams and hexagrams) that correspond to binary numbers, but they were not commonly used for computation.

B. In 4th-century-B.C.E. India, the poet and musician Pingala used a binary system to notate musical meters.

C. The modern binary system using 0s and 1s was established by the great German mathematician Gottfried Leibniz in the 1660s.

D. The first computer that used binary addition was built by mathematician George Stibitz, a scientist at Bell Labs, in 1937.

E. Not all of our numeral notation is positional even today.

1. We designate locations on the Earth's surface using latitude and longitude in degrees, minutes, and seconds.

2. Our division of time also shows the influence of the Babylonian base-60 system.

3. Stock prices traditionally are measured in eighths. The use of; this base may result from the practice of hand signaling in the trading pit.

Questions to Consider:

1. Why do you think it was difficult for so many cultures to consider zero a number?

2. Write the base-10 expansion for 2017. Write the base-2 expansion for 1101. Write the base-3 expansion for 2123. How would the last two numbers be expressed in the shorthand base-10 notation?

Lecture Five
The Magical and Spiritual Allure of Numbers

Scope Numbers were not only important tools in early civilizations, but they also were a source of mystery, entertainment, and even spirituality, Once humans named numbers, they became curious about studying numbers in their own right, We will look at some of the ideas of the Pythagoreans and other practitioners of number mysticism, Perfect numbers and amicable numbers —notions that originated in ancient Greece —continue to capture the imagination of individuals around the world today, Games involving numbers date back as early as 2000 B.C.E.: one of the most popular and persistent number challenges is the magic square, We will explore some of the fascinating appearances of magic squares throughout history, including their appearance in the work of Benjamin Franklin, Such number recreation draws humanity in to the exploration of many more serious and subtle properties of numbers.

Outline

I. Numbers appear in many rituals and beliefs.

A. In Babylonia, 60 was the number of Anu, the god of heaven, and 30 was the number of Sin, the lunar god.

B. Many modern religions specify the number of prayers to be recited.

C. In Islam, the number 5 is a good omen.

D. The number 4 is avoided in Japan because the word for 4. shi, sounds similar to the Japanese word for death.

E. In many cultures today, a well-known superstition surrounds the number 13.

II. In the 6th century B.C.E., Pythagoras— perhaps inspired by the Jains ushered in the dawn of a new era in how people viewed and studied numbers and mathematics.

A. The Pythagoreans elevated the study of numbers to the highest intellectual level.

1. They distinguished arithmetic (the study of numbers) from logistic (the practical use of numbers, or calculation).

2. They focused on the four disciplines (arithmetic, geometry, astronomy, and music); these four subjects formed the quadrivium, the basis of the liberal arts.

3. They studied numbers as abstract objects, exploring their intrinsic properties.

B. Pythagoras founded a community of scholars he called the Brotherhood.

1. Women were included as scholars in the Brotherhood, which was unusual for the time.

2. The Brotherhood believed that natural numbers were basic t all qualities of matter and living things.

3. They kept their studies secret, passing down beliefs and results in the oral tradition. Our knowledge of Pythagoras's work is based on the writings of scholars in later generation, including Euclid and Aristotle.

4. The community had many strict, peculiar rules.

C. The Pythagoreans believed in number characteristics.

1. The number 1, monad (unity), was the generator of all numbers and the number of reason.

2. The number 2. dyad (diversity, opinion) was the first female number.

3. The number 3, triad (harmony = diversity + unity) was the first male number.

4. The number 4 stood for justice or retribution, as in the "squaring of accounts" or "let's get this issue squared away.

5. The number 5 stood for marriage (2 + 3 = female + male).

6. The number 6 stood for creation (perhaps because 6 = 2 + 3 + 1).

7. The number l0. *tetractys*, was the holiest number and represented the four elements of the universe: fire, water, earth, and air, Geometrically, *tetractys* was represented by 1 dots arranged in an equilateral triangle; arithmetically, 10 = 1 + 2 + 3 + 4 (an example of a triangular number).

D. The Pythagoreans revered the relationship between certain ratio and musical harmonies, as well as aesthetic proportions.

1. They did not consider ratios-that is, tractions-to be actual numbers.
2. In their study of music, they observed that strings with lengths in a ratio of 1:2 vibrated in octaves, and strings with lengths in a ratio of 2:3 vibrated in a perfect fifth.
3. They were among the first to study the golden ratio, Two numbers are in the golden ratio if the ratio of their sum to the largest is equal to the ratio of the larger to the smaller, Using the quadratic formula, it can be shown that this golden ratio is $\frac{1+\sqrt{5}}{2}$ equal to the number = 1.6180339....
4. Pythagoras's wife, Theano, was a mathematician whose best work is said to have been on the golden ratio.

III. Mathematicians have explored figurate numbers, perfect numbers, and amicable numbers.

A. Figurate numbers are those that can be visualized in particular geometric ways, including triangles, squares, pentagons, and pyramids.

B. A perfect number is one that equals the sum of its proper divisors.

1. The number 6 is perfect because 6 = 1 + 2 + 3 and 28 is perfect because 28 = 1 + 2 + 4 + 7 + 14. The number 10. however, is not perfect because $10 \neq 1 + 2 + 5$.
2. Many open questions remain about perfect numbers: Is there an infinite number of perfect numbers? Are there any odd perfect numbers?

C. A pair of numbers is amicable if the proper divisors of the first sum to the second and vice versa.

1. The numbers 220 and 284 are amicable because the proper divisors of 220 are 1. 2. 4. 5. 10. 11. 70. 77. 44. 55. and 110. which sum to 284. and the proper divisors of 284 are 1. 2. 4. 71. and 142. which sum to 220.
2. Many open questions remain about amicable numbers: Is there an infinite number of pairs of amicable numbers? Are there any odd-even pairs of amicable numbers?

D. Perfect and amicable numbers are the subjects of some of the oldest unanswered questions in all of mathematics.

IV. Humans have played with numbers since at least 2000 B.C.E.

A. Dice made of fired clay dating from 3000 B.C.E. have been for in northern Iraq.

1. The ancient players did not yet have symbols for numbers, which is perhaps why they marked the sides of the dice with dots.
2. We use the same configuration on our modern dice today.

B. Ancient Egyptians played a finger counting game, Mora, as ea as 2000 B.C.E.

1. Each of two players extends his or her hand with any number of fingers showing.
2. Each player simultaneously calls out a number from 1 to the total number of fingers equals the number called, that player wins a point.

C. Magic squares have been a source of fascination, entertainment, and superstition for more than 3.000 years.

1. A Chinese book, Lo Shu (The Book of the River Lo), from l000 B.C.E. tells the story of a turtle emerging from the river with a pattern of dots carved into its back.
2. When written as digits, the pattern forms a 3 x 3 magic square:

4	9	2
3	5	7
8	1	6

3. A 3 x 3 magic square is a configuration in which the numbers 1 through 9 are each put in one cell so that the sum of numbers in each row, each column, and the two diagonals equals 15.
4. Analogously, *n x n* magic square is an arrangement of the numbers 1 through n2 into a square grid (each row contain n numbers) so that the sums of the numbers in each row, column, and main diagonal are equal.
5. The German artist Albrecht Dürer included a 4 x 4 magic square in his engraving Melancholia.

6. Magic squares were thought to protect against the plague.
7. Ben Franklin enjoyed the challenge of constructing magic squares while he was a clerk for the Pennsylvania Assembly.
8. Modern mathematicians have proved many results about magic squares, including existence theorems and results about related structures, such as Latin squares and orthogonal Latin squares.
9. Sudoku might appear to resemble magic squares, but there are no sums involved in Sudoku, just logic.

Questions to Consider:

1. Verify that 28 is a perfect number, Verify that 1,184 and 1,210 are amicable, (Hint: 37 is a factor of 1.184. and 11 is a factor of 1.210. twice!)
2. Complete the following magic square, (Hint: The sum is 34.)

	14		11
15			5
12		13	
			16

Lecture Six
Nature's Numbers—Patterns without People

Scope: Once our ancestors identified and named the abstract objects known as numbers, they gave birth to ideas that immediately took on a life of their own, attracting the interest and capturing the imagination of curious minds throughout human history, In this part of our course, we will explore the early steps of that historic journey to discover and appreciate the structure and beauty of numbers in their own right, We will start with the realization that numerical structure, beauty, and pattern existed long before humankind named the numbers, Our studies of fruit and flora will lead us to discover the famous Fibonacci numbers (a sequence of numbers that begins 1. 1. 2. 3. 5 ...), We will then return to the beautiful pattern that both nature and Fibonacci incorporated in their works and make a striking observation first established by Edouard Zeckendorf in 1972: Every natural number can be expressed uniquely as a sum of distinct Fibonacci numbers, We will see that every natural number, in some sense, is either a Fibonacci number or a not-so-distant cousin of several different Fibonacci numbers.

Outline

I. We begin our exploration into the nature of numbers by considering numbers in nature.

   A. Focusing on an ordinary pineapple, we notice two interlocking collections of spirals on the pineapple's façade and ask the nature question: How many spirals are there in each direction? We count and see 8 and 13 spirals and are surprised that the counts are not equal.

   B. We look at a daisy and a coneflower and are immediately drawn the spirals in their centers.

      1. We find 21 and 34 spirals, but we are no longer surprised that these values are different.
      2. If we count the spirals on the side of a small pinecone, we find 3 and 5 spirals.

   C. We now list the numbers we find in order and move from numbers in nature to the nature of numbers.

      1. We see: 3. 5. 8. 13. 21. 34.

2. If we add any two adjacent numbers on our list, the sum is the next number on our list.

3. If we generalize this pattern, then we could extend the list indefinitely: 3. 5. 8. 13. 21. 34. 55. 89. 144. 233 ....

4. We could even extend our pattern and our list of natural numbers in the other direction: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 ....

D. This abstract pattern can be used to better understand our physical world.

E. Why do nature's spiral counts so often appear on this special list of numbers?

1. The daisy is an example of a composite flower —each floret is an individual flower that grows from the stem and gradually moves outward toward the circular boundary.

2. Biologists hypothesize that these florets position themselves so that they have as much room around them as possible.

3. If we pack a circular disk with florets in this fashion, it can be proven that the spiral counts are always two consecutive numbers from the list we just discovered.

II. The history of these special numbers goes back more than two millennia.

A. This famous sequence was studied in India.

1. In the 3rd century B.C.E., Pingala included these numbers in his commentary on Sanskrit grammar.

2. In the 6th century C.E., the Indian mathematician Virahanka studied meter patterns of long and short syllables and showed how some meter counts corresponded to these numbers.

B. The numbers in this famous sequence are known as the Fibonacci numbers, named after Leonardo of Pisa, an Italian mathematician whom many believe was one of the greatest European mathematicians of the Middle Ages.

C. In 1202. Fibonacci published an influential text titled *Liber, Abaci* (The Book of Calculating) that introduced the Hindu-Arabic numerals to the West, He also strove to popularize the system.

D. His text explained how to perform arithmetic using the decimal system that we use today rather than the analogous calculations

using the more cumbersome Roman numerals, which were still popular in Italy at that time.

E. In chapter 12 of *Liber- Abbaci*, he posed a question aimed at calculating how many rabbits will be in an enclosed space after 12 months, given their reproductive habits.

F. We see that the number of pairs of rabbits present is equal to the Fibonacci numbers.

G. This solution was the first time this list of numbers appeared in the literature of Western mathematics.

III. The Fibonacci numbers continue to attract the attention of mathematicians and math enthusiasts.

A. Many results have been found about and involving the Fibonacci numbers.

B. Today, there is a research journal named The Fibonacci Quarterly

C. The Fibonacci numbers are connected to the golden ratio studied by the Pythagoreans.

1. If we consider the ratios of consecutive Fibonacci numbers-

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}$, and so forth— then these fractions approach the value of the golden ratio.

2. For example, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$ ...; and further out, $\frac{144}{89} = 1.6179 \ldots$

D. Though there are natural numbers that are not Fibonacci numbers. in 1972. Edouard Zeckendorf proved a beautiful result: If a natural number is not a Fibonacci number, then it can be written uniquely as a sum of nonadjacent Fibonacci numbers.

1. We notice that 4 = 3 + 1. 6 = 5 + 1, and 7 = 5 + 2.

2. This expression of natural numbers into Fibonacci numbers is now known as the Zeckendorf decomposition.

3. To express a natural number N in this manner, we apply a "divide and conquer" strategy, We first find the largest Fibonacci number less than the natural number N. then write as

the sum of that Fibonacci number and a remainder, We then repeat this process with the remainder, For example:

30 = 21 + 9 = 21 + 8 + 1.

4. This "divide and conquer" strategy can be used to prove this result in general.

Questions to Consider:

1. Consider two consecutive Fibonacci numbers, Square each and add the results together, Repeat this calculation with various consecutive Fibonacci numbers and see if you can find a pattern in your results.
2. Look for Fibonacci numbers in the world around you: Count the number of petals in flowers, the spirals in pinecones and pineapples, or the lobes on leaves.

Lecture Seven
Numbers of Prime Importance

Scope: Here, we will introduce the concept of prime numbers—those natural numbers greater than 1 that cannot be written as a product of two smaller natural numbers. The formal study of prime numbers dates back to the ancient Greeks, when Euclid first established the fact that there are infinitely many primes, considered by many to be one of the most elegant arguments in all of mathematics. We also will discover that prime numbers are the fundamental multiplicative building blocks for all natural numbers that exceed 1: that is, every natural number greater than 1 is either a prime number or can be expressed uniquely as a product of prime numbers. Although there appears to be no clear pattern to the prime numbers, great mathematical minds, including Carl Friedrich Gauss, wondered how many primes there are between 1 and any given number. The answer. now known as the prime number theorem, was given by Jacques Hadamard and Charles de la Vallee Poussin independently in 1896. The study of the primes remains an active area of research today, known as analytic number theory. We will conclude with several questions about the prime numbers that remain unanswered to this day.

Outline

I. What is a prime number?

A. A prime number is a natural number greater than 1 that cannot be written as a product of two smaller natural numbers.

B. The first few prime numbers are 2, 3, 5, 7,11, 13, 17, 19, and 23.

C. Why is 1 not a prime?

1. The primes that evenly divide a number reveal intrinsic features of that number (e.g., if the prime 2 evenly divides a particular number, that tells us that the number must be even).

2. The trivial fact that any number is itself multiplied by 1 (e.g., 6 = 1 x 6) leads to no new insights into that number. Therefore, 1 should not be viewed as a prime number.

3. The more theoretical explanation is that 1 is the *only* natural number whose reciprocal (namely, $\frac{1}{1}$) is also a natural

number. Because of this special property, 1 is called *unity*. In advanced mathematics, unity is never considered a prime.

II. Around 300 B.C.E., Euclid was the first to prove theorems involving prime numbers.

A. He was the author of 13 books known as the *Elements of Geometry*.

1. Many believe these to be some of the most important treatises ever written.

2. Euclid pulled many ideas together for the first time into a unified whole and established the notion of rigorous proof that remains with mathematics to this day.

B. Books VII. VIII. and IX of the *Elements* contained more than 100 theorems from number theory.

1. Euclid's Proposition 14 stated that every natural number greater than 1 can be expressed uniquely as a product of prime numbers (except for the rearrangement of the factors). For example:

$$12 = 2\times 2\times 3 = 2\times 3\times 2 .$$

2. This important result is now known as the "fundamental theorem of arithmetic."

3. We note that the uniqueness aspect of this result would no longer hold if we considered 1 a prime number

$$\left(6 = 2 \times 3 = 1 \times 2\times 3\right).$$

C. Proposition 20 of Book IX states that prime numbers are more than any assigned multitude of prime numbers.

1. Today, we would rephrase Euclid's result by stating that there are infinitely many primes.

2. Euclid's proof is considered by most mathematicians today as one of the most elegant proofs in mathematics.

D. In order to inspire the beautiful idea in Euclid's proof, we explore how to use the primes 2 and 3 to find a third prime.

1. We wish to create a number that is not a multiple of 2 or 3. 2. We first multiply the numbers 2 and 3 together. then add 1:

2 x 3 + 1.

2. We note that 2 x 3 + **1 =** 7, a number that has neither 2 nor 3 as a factor; thus, there must exist a third prime (in this case, 7 itself happens to be a prime).

E. Using the previous idea, we prove Euclid's theorem in general.

1. Suppose we have a finite list of primes (2, 3, 5, 7, ..., p, with p denoting the last prime number on our list). and we wish to show that there exists a prime number that is not on our list.

2. We consider the number (2 x 3 x 5 x 7 x … x *p)* + 1 and notice that none of the primes on our list can divide evenly into this number.

3. By the fundamental theorem of arithmetic there must exist a prime that divides this number that was not on our original list.

4. Because this argument can be applied to any finite list of primes, we conclude that there are infinitely many primes.

III. Our understanding of the primes has advanced since Euclid's time.

A. How many primes are there up to a certain point?

1. We know by Euclid's theorem that there are infinitely many primes. Is there a way, however, to know how many primes there are up to any particular natural number n?

2. We write P(n) to represent the number of prime numbers less than or equal to *n.*

3. For example, P(5) = 3 because there are three primes less than or equal to 5 (2. 3, 5). Similarly, P(20) = 8 (the eight primes are 2, 3, 5, 7, 11. 13, 17, 19).

4. Is there a formula for P(n) in general? This question remains unanswered to this day. Many mathematicians throughout the ages contemplated this question, including Carl Friedrich Gauss (1777-1855), whom many consider the "Prince of Mathematics."

B. The prime number theorem can be explained as follows.

1. Many individuals (including Gauss) noticed that the number P(n) was closely approximated by ; In(n) is known as ln(n) the natural logarithm. and its value can be found on any scientific calculator.

2. This result was finally proven to hold by Jacques Hadamard and Charles de la Vallee Poussin independently in 1896.

3. The prime number theorem implies that as *n* gets larger and larger. P(n) gets closer and closer to $\frac{n}{l\,n(n)}$. More precisely, as n gets larger and larger, $\frac{P(n)}{\frac{n}{\ln(n)}}$ approaches 1.

4. Because the ratio above *approaches* 1. there is always some ever-shrinking gap between that ratio and 1. That gap can be viewed as an "error." Exploring how the "error" in the previous result is shrinking is a very active field of study (an important part of analytic number theory) and is connected to the famous open question known as the Riemann Hypothesis.

IV. Finding the largest prime known to date is an ongoing quest that has captured the imagination of thousands of people and computers.

A. The largest prime known today was found in September 2006 using a number of supercomputers.

1. It is $2^{32582657} - 1$ and has 9,808.358 digits: printing it out in 12-point font would require more than 2.500 pages.

2. This large prime is an example of a Mersenne prime (a prime of the form 2" - 1); this is the typical form of the large primes that are found today.

B. Beyond the prime 2, the closest two adjacent primes can be to each other is two numbers. Two primes whose difference is 2 are called twin primes.

1. For example, (3. 5). (5, 7), (11, 13) are twin primes, while (7, 11) are not twin primes.

2. The Twin Prime Conjecture states that there are infinitely many twin primes. Although many mathematicians believe that this statement is true, no one has yet produced a complete proof of it.

C. Can every even natural number greater than 4 be expressed as the sum of two odd prime numbers? This is known as the Goldbach conjecture.

1. Notice that 6 = 3 +3 , 8 = 3 +5 , 24 = 5 +19 , and even 1.000 =

3 +997 .

2. This question, first stated in a letter by the Prussian mathematician Christian Goldbach to the Swiss mathematician Leonhard Euler on June 7, 1742, is one of the oldest unanswered questions in mathematics today.

3. It is known that the conjecture holds for all even numbers up

   a. to 3×10^{17} .

4. There is a $1 million prize for the first correct and complete proof of this conjecture.

Questions to Consider:

1. Apply the strategy Euclid used to prove there are infinitely many primes to provide an argument for why there must be a prime greater than 1,000.000.

2. Prime numbers can be viewed as the building blocks, or "atoms," of the natural numbers. What other structures or systems in our lives have atoms?

Lecture Eight
Challenging the Rationality of Numbers

Scope: When we count wholes that are composed of smaller parts, we run into trouble when we count 1, 2, 3 ... because we may have a quantity that is more than 2 but less than 3. This counting conundrum was resolved early in human history by extending the notion of number to include fractional parts (i.e., rational numbers). Both Babylonians and Egyptians used fractions perhaps as early as 2000 B.C.E. Although the Pythagoreans did not consider these ratios to be God-given numbers, they were convinced that all lengths could be measured in terms of the natural numbers and these ratios. The Pythagoreans were forced to let go of this aesthetically appealing notion of number when they discovered lengths whose measure could not be a rational number. This unattractive reality further challenged the Pythagorean notion of number—in fact, Pythagoreans did not consider these measures to be numbers. Today, we call these very real numbers that are not rational numbers *irrational numbers—literally,* "numbers without ratios." The discovery of such perplexing quantities brought to light the surprising subtlety and delicate structure within the world of number when viewed as lengths and measures. Embracing this expanded view of number is a great challenge for all who face it for the first time.

Outline

I. What are rational numbers, and when are they recognized as such?

A. A rational number is a ratio of two integers $\frac{m}{n}$, for which n is positive.

1. The number m is called the numerator and n is called the denominator.
2. Rational numbers are also known as fractions.

B. Although we do not know the first example of rational numbers, there is evidence of fractions in Babylonian and Egyptian writings from around 2000 B.C.E.

1. §A unit fraction is defined to be any fraction having its

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numerator equal to 1 (e.g., $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$), the Rhind Papyrus contains sums of unit fractions.

2. Ahmes, the scribe who wrote the Rhind Papyrus denoted unit fractions by a dot over the denominator. For example, he would write a dot over the number 5 for the unit fraction $\frac{1}{5}$. This dot notation evolved from the hieroglyph for an open mouth.
3. This symbol offers evidence that fractions were used to divide rations of food and drink. Many questions from the Rhind Papyrus concerned dividing loaves of bread and jugs of ale.
4. In addition, the Rhind Papyrus contains a table that shows how, for any odd natural number n from 5 to 101, fractions of the form $\frac{2}{n}$ can be written as the sum of different unit fractions (e.g., $\frac{2}{5}$ can be written as $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$.
5. An Egyptian fraction is the sum of distinct unit fractions. It is not clear why the Egyptians chose this method of expressing fractions.

Fibonacci showed that every positive rational number can be expressed as an Egyptian fraction.

II. The ancient Greeks—in particular, the Pythagoreans—were one of the first cultures to explore rational numbers as objects of independent, abstract interest.

A. The ancient Greeks had a complicated view of number. They believed that the natural numbers were the true numbers because they were God-given.

1. The rational numbers were really only ratios of natural numbers—they were not as real as the natural numbers.
2. In the eyes of the Greeks, however, the rational numbers formed a gap-free list that could correspond to a line of numbers; the Pythagoreans viewed these numbers as a "flow of quantity."

B. Euclid made the Greek notion of rational numbers much more rigorous.

1. In *Elements,* he declared that two line segments *A* and *B* are commensurable if there exists a third line segment *C* such that some natural number copies of *C,* when laid end-to-end produce the segment *A* (and similarly for the segment *B).*

2. If we call the lengths of these line segments *a*, *h*, and *c*, respectively, then we see that there exist natural numbers *m* and *a* such that $a = cm$ and $b = cn$.

3. We observe that the ratio of *a* to *h* is as follows:

$$\frac{a}{b} = \frac{cm}{cn} = \frac{m}{n}.$$

4. We say that two numbers *a* and *b* are commensurable if $\frac{a}{b}$ is a rational number.

5. The Greeks believed that every two lengths (numbers) were commensurable: thus, the Greeks believed that all lengths were either natural numbers or ratios of natural numbers—what we today call rational numbers.

C. A certain problem revealed the fallacy in this thinking.

1. Suppose we consider a square having a side length of 1 unit.

2. If we draw a diagonal in that square, then we can measure its length.

3. In view of the Pythagoreans' belief that all length numbers are rational, the length of the diagonal of our square must be a rational number.

III. The Pythagoreans made a surprising discovery.

A. Labeling the lengths of our line segments leads us to their discovery.

1. We let h represent the length of the diagonal of the square.

2. Given the Greeks' belief. *h* must be a rational number (let us call it $\frac{m}{n}$ for some natural numbers *m* and *n.*

3. Thus, $h = \frac{m}{n}$.

B. The Pythagorean Theorem states that the sum of the squares of the lengths of the legs of any right triangle equals the square of the length of the remaining side (the hypotenuse).

1. In our right triangle, we see that $1^2 + 1^2 = h^2$, or equivalently,

$$2 = \left(\frac{m}{n}\right)^2 \text{ thus, we have } 2 = \frac{m^2}{n^2}.$$

2. Solving for m we find that $\sqrt{2} = \frac{m}{n}$; that is, , $\sqrt{2}$ must be a rational number. Therefore, the natural numbers *n* and *m* satisfy the equality: $2n^2 = m^2$

3. By Euclid's Proposition 14, we know that every natural number greater than 1 can be uniquely written as a product of prime numbers. Also, because we are squaring *m* and *n*, every prime factor of m^2 and of n^2 must appear an even number of times.

4. There are an odd number of factors of 2 in the number 2n2; thus, there are an odd number of factors of 2 in the number m which is a contradiction.

5. The only assertion that was made without rigorous proof was the Greek belief that all numbers are rational. We now see that this assumption led us to a contradiction.

6. We have constructed a line segment whose length is not a "number" according to the Greek notion of number as rational.

7. This argument showing that $\sqrt{2}$: is not a rational number—similar to Euclid's original proof—is considered by many to be one of the most elegant proofs in mathematics.

IV. The dawn of irrational numbers came despite the aesthetically appealing ideal of a rational world of numbers.

A. We now know that we can find lengths that are not rational.

1. Intuitively, it is clear that the length of any line segment should be a number.

2. We are, thus, at an impasse. The length of our diagonal, $\sqrt{2}$, is not a number (because it is not rational), yet $\sqrt{2}$ does represent a length and all lengths should be numbers.

B. For this reason we must expand the Greek notion of what *number* means so that it now includes values that are not rational numbers.

1. Irrational numbers are numbers that are not expressible as a ratio of two integers; that is, numbers that are not fractions.
2. Sometime between 2000 and 1650 B.C.E., the Babylonians computed rational approximations to $\sqrt{2}$, but the Greeks were the first to prove it as irrational.

C. It is suspected that the Secret Society of Pythagoreans reacted skeptically.

1. The Pythagoreans were perplexed by the irrationality of $\sqrt{2}$.

   Some scholars believe that they did not accept $\sqrt{2}$ as a number.
2. The Pythagoreans called irrational numbers *alogos*, which translates into "unspeakable" or "inexpressible."
3. There are many legends and theories about how the Pythagoreans interpreted this discovery and how they reacted to it. Although we will probably never know their view of this counterintuitive fact for certain, it is clear that this discovery moved our understanding of number forward.
4. Around 370 B.C.E., the Greek astronomer and mathematician Eudoxus, who was a student of Plato, offered a definition of irrational numbers that beautifully foreshadowed the work of mathematicians Karl Weierstrass, Georg Cantor, and Richard Dedekind in Germany in the 19th century.

Questions to Consider:

1. Can you modify Euclid's proof that $\sqrt{2}$ is irrational to show that $\sqrt{3}$ is irrational? *Bonus Challenge:* Work through Euclid's argument with $\sqrt{4}$ and determine at what point the proof fails to yield a contradiction.

   (This is a good thing, because we recall that $\sqrt{4}$ = 2 is indeed, a rational number.)

2. The term *rational* derives naturally from the word *ratio,* and the term *irrational* derives naturally,„ from *not rational,* yet the word *irrational* has a nonmathematical meaning, as well. What impact do you believe this other meaning may have on how people perceive irrational numbers?

Lecture Nine
Walk the (Number) Line

Scope: In order for us to develop some intuition into the vast collection of real numbers, we will consider the visualization of this collection through the real number line. This geometric point of view provides a very useful description of the collection of real numbers: Each point on the line corresponds to a real number according to its distance from the point marked 0 and whether it is to the right or left of O. It also provides a means of expressing a real number as a decimal expansion. We will look at how to express real numbers in other bases. Studying how decimal expansions for rational numbers eventually become periodic will lead us to some counterintuitive discoveries. We will see that the number line, together with decimal expansions, offers an attractive and powerful means of expressing and better understanding real numbers.

Outline

I. How do we represent the real numbers on a line?

A. The term "real numbers" itself is misleading and was probably coined in response to the discovery of yet more abstract imaginary numbers.

B. One view the Pythagoreans held for number was as a "flowing quantity." Even though they were unable to accept irrational numbers, their intuition about a "flowing" list of numbers was correct. We now view this flow in the form of a line.

1. Flemish mathematician Simon Stevin (1548-1620) may have been the first to consider the number line in a text he authored in 1585, which introduced a systematic approach to the arithmetic of decimal numbers.

2. In 1637, Rene Descartes described the "Cartesian coordinate system" that implicitly utilized the number line.

3. Zero takes center stage as we move away from the Greek sense of number.

4. We have a *continuum* of numbers. This concept forms an important connection between points on a line and a notion of number.

II. Stevin was the first to offer a thorough account of decimal expansions of real numbers.

A. For our purposes here, we will define the real numbers as the collection of all numbers that can be written as an endless decimal expression (e.g., 3 = 3.000…, 7.5 = 7.5000..., $\frac{1}{3}$ = 0.3333..., $\sqrt{2}$ = 1.414213..., –4.012 = –4.0 12000._ and π = 3.1415... are all examples of real numbers, each expressed as an endless decimal expansion).

B. Stevin's original notation foreshadowed our current decimal notation.

1. Stevin's system expressed $61\frac{837}{1000}$ as $61\ 8^{'}\ 3^{''}\ 7^{'''}$.

2. By 1653 the notation had evolved so that this number would have been written as 61:837: today, we would write 61.837.

C. We can locate a real number, such as 1.417, on the number line by repeatedly dividing the line into 10 equal-length intervals.

D. Simon Stevin saw the practical value of the decimal system and argued that the monetary system should be decimal-based rather than based in fractions.

III. We explore how to express real numbers in other bases.

A. We start with binary expansions.

1. There is nothing particularly special about the decimal expansion *(deci* implies a base of 10).

2. We can consider representing real numbers in other bases.

3. This representation would be equivalent to repeatedly dividing whole-number intervals into another number of equal-length sub-intervals.

4. The simplest base is base 2—giving us what are known as binary expansions.

5. The 17th-century German mathematician Gottfried Leibniz, who invented calculus independently and concurrently with Isaac Newton, believed that binary expansions were extremely valuable and even held quasi-mystical properties.

B. The next most basic expansion is the base-3 expansion—also known as the ternary expansion of real numbers.

1. We divide the whole-number intervals of the number line into three equal-length subintervals, then repeat.

2. The 17th-century French mathematician Blaise Pascal believed that there was nothing innately special about base-10 representations—any other base, such as binary or ternary, was just as "natural."

IV. Long division reveals the decimal expansions.

A. Decimal expansions for rational numbers must eventually become periodic—that is, eventually, they must have an endless "tail" of repeated digits in the decimal expansion.

B. Periodic decimal expansions lead us to other revelations.

1. Revisiting 0.3333... leads us to the converse result: All periodic decimals are rational numbers.

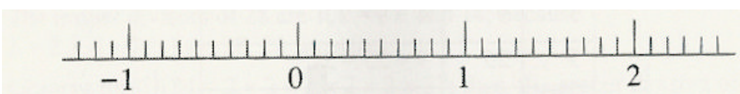
2. We find a surprise hidden in the number 0.9999... as well as in other bases.

3. We can classify the rational numbers as those numbers whose decimal expansions are eventually periodic.

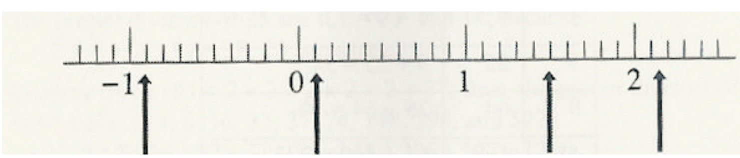
4. This result captures, in spirit, the "finite" nature of rational numbers.

§Questions to Consider:

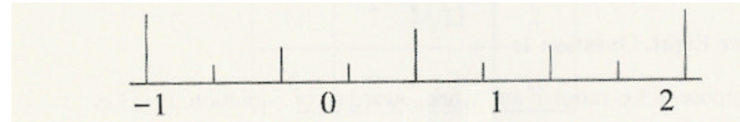
1(a). Match the decimal numbers 1.5, 0.1, 2.15, and -0.9 to their respective places on the number line below.



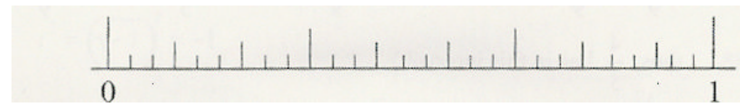
(b) For each point marked with an arrow on the number line below, write the corresponding decimal number it represents



2(a). Below is a portion of the binary number line, with hash marks to show halves of intervals instead of tenths. Locate the binary numbers 0.1_2, 0.01_2 and 0.101_2, on this number line.



(b). Below is a portion of the ternary number line, with hash marks to show thirds of intervals instead of halves or tenths. Locate the ternary numbers 0.11_3, 0.22_3, and 0.002_3 on this number line.



Lecture Ten
The Commonplace Chaos among Real Numbers

Scope: The decimal representation of real numbers offers an alternative means of classifying the enigmatic irrational numbers. This classification provides us with an attractive means of producing other irrational numbers. In view of the fact that real numbers are either rational or irrational, a natural question emerges: What proportion of real numbers is rational and what proportion is irrational? Our classification of rational and irrational real numbers via their decimal expansions holds the key to unlocking an intuitive answer to this question. It also leads to a surprising realization that we will make mathematically precise: No real numbers are rational and all real numbers are irrational! This assertion is counterintuitive because the rational numbers are the ones with which we are most familiar and the irrational numbers appear to us as exotic and unusual. Here, we discover an important realization that will become a recurring theme: Often, the seemingly familiar is the exception and the exotic is the norm. Although the rational numbers are a sparse set of real numbers, we will see that they are spread throughout the number line and are dense. We close this lecture with a description of some incredible facts involving the related modern concept of normal numbers, first studied by the important French mathematician Emile Borel in 1909.

Outline

I. The rational numbers are precisely those real numbers whose decimal expansions are eventually periodic.

A. The decimal expansion for $\sqrt{2}$ will never become periodic because it is irrational.

B. We can describe "new" irrational numbers via their decimal expansions (e.g., the number 0.101001000100001000001... must be irrational because the decimal expansion never becomes periodic).

II. Real numbers come in two "flavors"—rational and irrational.

A. Rational numbers are the ones that are the most familiar to us because we use fractions regularly in our daily lives.

B. Irrational numbers are far more exotic; they are difficult to name.

and it is challenging to verify their irrationality.

C. Suppose we pick a number at random. The random real number will either be a rational number or an irrational number.

D. What is the probability that a random real number is rational?

1. We can produce a random number by generating its digits through chance.

2. Rolling a 10-sided die will generate random digits. The likelihood that a real number selected at random is rational is 0%, while the likelihood that a real number selected at random is irrational is 100%!

E. There are, in some sense, "more" irrational numbers than rational ones.

1. Numbers that first appeared to be exotic are more the norm, and numbers that first appeared to be the norm are, in actuality, exotic.

2. Our everyday experiences with numbers do not allow us to gain an accurate sense of how the rational and irrational numbers "fit" together.

III. Given any two distinct points on the number line (that is, any two unequal real numbers), there is a rational number between them.

A. Our method to establish this fact goes back to the ideas of the great Greek mathematician Archimedes.

B. Because the rational numbers exhibit this property, we say that they are "dense" on the number line (or, equivalently, they are dense in the real numbers).

IV. We look at how to reline our notion of a random real number.

A. If we consider rolling a 10-sided die to generate digits of random real numbers in base 10, how often would we expect the digit 3 to appear in a random real number?

1. We would guess that every digit from 0 to 9 would appear an equal amount of the time (on average); that is, each digit would appear approximately $\frac{1}{10}$ of the time.

2. If we select a real number at random, we expect that $\frac{1}{10}$ of

the digits in its decimal expansion will be 0, $\frac{1}{10}$ of the digits will be 1, and so forth to the digit 9.

B. Normal numbers in base 10 are explained.

1. We might guess that because there are 100 two-digit numbers (from 00 to 99), each two-digit number will appear approximately $\frac{1}{100}$ of the time in the decimal expansion of an average real number.

2. Real numbers for which every length of digits appears the "expected" amount of time in their decimal expansions are called "normal numbers in base 10."

3. In 1909, the French mathematician Emile Borel proved that the likelihood that a random real number is normal in base 10 is 100%.

4. In view of our previous observations, we see yet again through this refined notion that rational numbers, such as 1 3 and $\frac{1}{}$, are very strange indeed.

C. Borel considered an even more general notion of random numbers.

1. A "normal number" is defined to be a number that is normal in every base (that is, normal in base 2, 3, 4, 5 ...).

2. Borel then proved an amazing theorem: The likelihood that a random real number is a normal number is 100%.

D. "Almost all" real numbers are normal.

1. lithe likelihood that a random real number possesses a particular property is 100%, then we say that "almost all" real numbers possess this particular property.

2. We have seen that "almost all" numbers are irrational and, furthermore, that "almost all" numbers are normal (hence the name).

3. For any random real number, we expect the digits to be such that any particular run of digits will occur its fair share of the time.

Questions to Consider:

1. Assuming that any obvious pattern continues, which of the numbers below are rational and which are irrational?

0.10110111011110...

3.7878787878...

0.123456789101112... (This is known as *Mahler's number.)*

-81.41783059833167098312... (a number selected at random).

2. When someone says, "Pick a number between 1 and 4," we always choose a whole number, which is certainly rational. Why does this happen? What effect do assumptions about context have on probabilities? *(Fun Fact* When we ask people to pick a number between 1 and 4, most people pick 3—try it!)

Lecture Eleven
A Beautiful Dusting of Zeroes and Twos

Scope: In this lecture, we explore and exploit the beautiful visualization of the base-3 expansion of the real numbers by studying one of the most famous and vexing collections of real numbers, described by the German mathematician Georg Cantor in 1883 Known as the "Cantor set" or as "Cantor dust." this collection of numbers is intriguing in its own right because the numbers can be described in a very elegant fashion in terms of their base-3 expansions or, equivalently, in terms of their location on the number line In fact, when viewed on the line, we can see that this collection is an example of what mathematicians call a fractal—a geometric form that has an endless self-similarity property We will discover that even though there are infinitely many numbers in the Cantor set, they form a very sparse collection of numbers between 0 and 1 on the number line. As sparse as this collection is, we will establish a truly remarkable and counterintuitive theorem: Even though this Cantor set is so dust-like, it contains just as many numbers as the entire line segment from 0 to 1. This result, and the clever argument that establishes its validity, shows us that within the jungle of the real numbers, many surprising and counterintuitive phenomena can and do occur. This lecture foreshadows the original, imaginative, and groundbreaking work of a great mathematician who set the mathematics community on fire with his ideas.

Outline

I. Georg Cantor lived in an exciting age for mathematics.

A. He was born in 1845 in St. Petersburg; soon after, his family moved to Germany, where he spent the rest of his life.

1. As a youth, Cantor showed much interest and talent in art, music, and mathematics.

2. In 1866, at the age of 22, Cantor completed his studies in number theory at the University of Berlin after studying under some of the greatest German mathematicians of the time: Ernst Kummer, Leopold Kronecker, and Karl Weierstrass.

B. In the mid-1800s there was much interest in subtle and delicate questions involving numbers and functions.

1. A critical question regarding the computation of interest was

how to express exotic functions as sums of more familiar trigonometric functions (such as the famous sine and cosine waves we see in visualizations of sound waves). Joseph Fourier the 19^{th}-century French physicist, developed a famous theory—Fourier analysis—in response to this question.

2. By 1870, it was known that the functions of interest to the mathematicians of the day could indeed be expressed as a (possibly endless) sum of sine and cosine functions. The big question, however, was whether such an expression was unique—that is, is there only one way to write these functions as sums of trigonometric functions?

3. In 1868, Cantor was offered a position at the University of Halle outside Berlin. Two years later, one of his senior colleagues, Edward Heine, published a paper in which he mentioned this question of uniqueness, noting that many great minds, including Rieman", had been unable to solve this important question.

4. This challenge captured Cantor's imagination and he set off to tackle this question that the great minds of the day could not answer.

C. By 1871, Cantor published an answer to the uniqueness question.

1. He proved that if the functions were reasonably "nice" and "well-behaved" (as most functions of the day were), then there is only one way to express them as a sum of trigonometric functions.

2. Cantor soon proved that, in fact, the functions could be "bad" as long as they were "bad" at only finitely many numbers; this result was well received.

3. Cantor began to consider functions that were "bad" at *infinitely* many numbers and extended his uniqueness result to such functions as long as these "bad" numbers were spread out; this result turned some heads.

4. He then wondered what would happen if the functions were "bad" at infinitely many numbers (or, equivalently, at infinitely many points on the number line) that were clustered together; it was at this point that Cantor's work became groundbreaking and totally original.

D. In order to study such delicate infinite clusters of points on the

number line, Cantor needed a precise definition of real numbers and a more rigorous treatment of irrational numbers.

1. Cantor started studying exotic collections of real numbers, and soon, this study captured his imagination.

2. His work was highly controversial at first and he suffered greatly; however, in the scope of just five years, his work moved the frontiers of mathematical and numerical understanding out beyond anyone's wildest imagination.

II. Recall the ternary expansion of real numbers.

A. If we look at the real numbers in base 3 (as Cantor did in 1883), rather than base 10, the collection we will explore will reveal itself much more naturally.

1. We recall that every real number can be expressed in a base-3 expansion.

2. In a base-3 expansion, the only allowable digits are 0, 1, 2; the "positional values" are powers of 3.

3. For example:

$120_3 = (1 \times 9) + (2 \times 3) + (0 \times 1)$

$= 1 \times 3^2) + (2 \times 3') + (0 \times 3°)$

$= 15$

$$0.1021_3 = \left(1\times\frac{1}{3}^1\right)+\left(0\times\frac{1}{3}^2\right)+\left(2\times\frac{1}{3}^3\right)+\left(1\times\frac{1}{3}^4\right)$$

$$= \frac{34}{81} = 0.419753086..._{10}.$$

B. We can visualize base-3 expansions on the number line by cutting up intervals into three equal pieces.

III. We will consider all real numbers within the interval between 0 and 1 that can be expressed in a base-3 expansion using only the digits 0 and 1

A. We will call this collection of numbers the Cantor set or Cantor dust and denote the collection by C.

B. Some examples follow.

1. The numbers 0 and 1 are in this collection (recall that

$1 = 0.2222_3$.

2. The numbers $\frac{1}{3}$ and $\frac{2}{3}$ are in this collection because

$\frac{1}{3} = 0.1_3 = 0.0222222\ldots_3$, and $\frac{2}{3} = 0.2000000..._3$.

3. Unfamiliar numbers, such as $0.220222000020202022202002222_3$ are also in this collection. In fact, there are both rational and irrational numbers in the Cantor dust.

4. This collection C appears to have no apparent structure.

IV. We find points in the Cantor set on the number line.

A. We plot the points 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1 on the number line and attempt to pinpoint $0.220222000020202022202002222..._3$.

B. We discover that the numbers in the collection C reside in either the first $\frac{1}{3}$ segment or the third $\frac{1}{3}$ segment of the interval between 0 and ; the collection C contains no points from the middle $\frac{1}{3}$ of the interval.

1. Within each of these two line segments of length $\frac{1}{3}$ the numbers from C.reside in either the first or third $\frac{1}{3}$ of these segments; we essentially remove the middle $\frac{1}{3}$ of each of these smaller intervals.

2. This continual pruning process leads to a geometrical description of this collection of numbers: The Cantor set contains all the numbers (points) that have not been removed. Given this description, the Cantor set is often called the "middle thirds Cantor set."

C. Cantor extended this notion and defined more general collections of numbers.

This geometric description shows that these numbers have some amazing structure—in particular, this collection has a self-similarity property that makes it an example of a fractal collection of numbers.

Fractals are images that exhibit a self-similarity property as we zoom in and focus on just a small part of the entire image.

V. No number in the Cantor set is normal.

A. We recall that normal numbers have base-3 expansions in which each of the digits 0, 1, and 2 occurs one-third of the time.

B. The numbers in the Cantor set do not contain the digit 1 when expressed in base-3 expansions; thus, none of the numbers of the Cantor set are normal.

C. "Almost all" real numbers are normal; thus, we conclude that "almost no" numbers are in the Cantor set.

1. If we were to pick a random real number between 0 and 1, the probability that we happen to select a number from the Cantor set is 0%; thus, we say that "almost no" numbers are in the Cantor set.

2. In some real sense, the Cantor set is a very sparse collection of numbers. This is why we also call this collection Cantor dust—it is essentially invisible on the line segment between 0 and 1.

VI. Even though "almost no" numbers are in the Cantor set, there is a one-to-one pairing of the numbers in the Cantor set with *all* the numbers on the real line between 0 and 1.

A. Cantor proved that even though his set is an essentially invisible collection of numbers inside the interval from 0 to 1 on the real line, that collection of numbers can be placed in a one-to-one correspondence as the entirety of numbers inside the interval from 0 to 1. Cantor challenged Euclid's notion that the whole is greater than the part.

B. The proof for Cantor's claim is elegant.

1. We must find a way of associating each number in the Cantor set with only one number the from the line segment between 0

and 1, and, conversely, find a way of associating each number from the line segment between 0 and 1 with one and only one number from the Cantor set.

2. We must remember that every real number can be expressed in a base-2 expansion using the digits 0 and1thus, every number from the line segment between 0 and 1 can be expressed as an endless list of Os and 1s.

C. To establish that every number from the Cantor set can be paired up with exactly one number between 0 and 1, consider a number from the Cantor set (recall that we express these numbers in base 3):

0.22000020202222200020...$_3$.

1. Change each 2 to a 1, and now view that number in base 2:

0.11000010101111100010_,

This new base-2 number is in the interval between 0 and 1.

2. Pair every number (expressed in base 3) in the Cantor set with a corresponding number (expressed in base 2) in the interval between 0 and 1 by simply changing all the digits that are 2s to 1 s and viewing the new list of digits as a base-2 number.

D. Why are there no numbers in the interval between 0 and 1 that have not been paired with any number from the Cantor set in this manner?

1. Consider the number 0.01011010100010101101_, We can find a number in the Cantor set that is paired with this particular number by simply performing the previous steps in reverse—that is, replacing all the 1s with 2s and viewing the number in base 3. In this case we would see the number 0.02022020202000202022202...3.

2. Recall that the Cantor set contains precisely all numbers between 0 and 1 whose base-3 expansions consist only of the digits 0 and 2; thus, the previous number is in the Cantor set and is paired up with the original number we considered from the interval 0 to 1 .

3. We now have found a one-to-one pairing between the Cantor set and the entire collection of numbers between 0 and 1, which was what Cantor claimed.

E. The Cantor set is a very counterintuitive collection of numbers and is often used as a counterexample to seemingly reasonable

conjectures.

Questions to Consider:

1. Which of the following numbers—$\frac{1}{2}$, $\frac{4}{5}$, $\frac{1}{3}$ and 0—are in the Cantor set? What about the following numbers written in base 3: 0.202_3; and 0.121_3?
2. The Cantor set can be constructed by repeatedly removing "middle thirds" of intervals starting with the interval between 0 and 1. What kind of set would we get if we started with the interval between 0 and 1 and successively removed middle fifths? Can you draw this collection on a number line?

Lecture Twelve
An Intuitive Sojourn into Arithmetic

Scope: This lecture opens with a brief historical overview of the familiar arithmetic functions of addition, subtraction, multiplication, and exponentiation, as well as the axioms that accompany them. Many of the basic rules of arithmetic are so obvious and sensible that we accept them without a second thought. One "rule" that we learned in our youth that does not fit in this category is the declaration that a negative number multiplied by another negative number produces a positive number. Here, we will discover that this "rule" is, in fact, a theorem that can be deduced in an attractive manner from the basic axioms of arithmetic: thus, we finally resolve a conundrum that has mystified many individuals since early childhood. Before moving on, we will explore the origins and evolution of the symbols we know so well. These include +, x, and ±. Combining numbers using these simple operations leads to some interesting and important numerical questions; for example, what does $2^{\sqrt{2}}$ mean, and is it a number? We will put the ideas of Nicholas Oresme (c. 1320-1382) and others together to make sense of such an unusual object and. moreover, to realize that such a thing is, indeed, a number.

Outline

I. The axioms of combining numbers are included in most of the fundamental rules of arithmetic that we still apply today.

   A. In the 3rd century B.C.E., Euclid offered five "common notions"—basic and self-evident truths that did not require proof. Today, we call such statements "axioms."

      1. Things that are equal to the same thing are also equal to one another. (Today, we call this property "transitivity.")
      2. If equals be added to equals, the wholes are equal.
      3. If equals be subtracted from equals, the remainders are equal.
      4. Things that coincide with one another are equal to one another. (Today in geometry, we say they are "congruent.")
      5. The whole is greater than the part. (Note that this notion is not always true.)

B. The numbers 0 and 1 have special properties.

1. The number 0 is known as the additive identity because if 0 is added to any given number, the sum is that given number.
2. The number 1 is known as the multiplicative identity because if 1 is multiplied by any given number, the product is that given number.

C. A review of inverses and operations is useful at this point.

1. Given any number *n*, there exists a number -*n* such that its sum with *n* equals 0. The number -*n* is called the –'additive inverse" of *n*.
2. Given any nonzero number *a*, there exists a number .[1] such $\frac{1}{n}$ that its product with *a* equals 1. The number $\frac{1}{n}$ is called the "multiplicative inverse" of *n*.
3. In an arithmetic sense, addition and subtraction are opposite operations, just as multiplication and division are. This observation was first made by the ancient Egyptians, who viewed the pairs of operations as mirror images of each other. Today, we say they are "inverse operations."
4. Zero is the only number that does not have a multiplicative inverse because when any number *a* is multiplied by 0, the product equals 0. This statement is not an axiom but a provable theorem.
5. Numbers also satisfy the distributive property:

   $a\times(b\pm c) = a\times b\pm a\times c$.

6. Thus, for any number *n*, we see that n

   $n\times 0 = n\times(1-1) = n\times 1 - n\times 1 = n - n = 0$.

II. Why does a negative multiplied by a negative equal a positive?

A. Multiplying signed numbers follows simple rules.

1. A positive number multiplied by a positive number is positive.
2. A negative number multiplied by a positive number is negative.
3. A negative number multiplied bya negative number is

positive.

B. We can prove that (-1) x (-1) = 1.

1. We first recall that because -1 is the additive inverse of 1, we have: 1 + (-1) = 0.
2. If we multiply both sides by (-1), we see by the distributive property that: (-1) x (I) ± (-1) x (-1) = (-1) x (0).
3. Recall that 0 multiplied by any number is 0. Also, 1 is the multiplicative identity; thus, we conclude:

   $-1+(-1)\times(-1)=0$

4. Adding 1 to both sides reveals: (-1) x (-1) = 1

III. The symbols of arithmetic evolved from numerous sources.

A. Addition and subtraction symbols seem to have derived from Latin.

1. In ancient Latin manuscripts, the word *et ("and")* was used to indicate addition. It appears that the symbol ÷ derives from the t in *et.*
2. In a manuscript from 1489, we find the Greek cross used for addition. The Latin cross was also used but printed horizontally. Sometimes, the Maltese cross was used for very formal writings.
3. The minus sign (-) made its first appearance in 1481.

B. Multiplication and division signs have multiple origins.

1. The multiplication sign (x) first appeared in Fibonacci's *Liber Abbaci.* He employed the notation to indicate what we might consider today as "cross products" of' pairs of numbers. Slowly, the notion evolved into the product of two numbers.
2. In 1751, an inverted D was used for division. By 1753, we find the symbol
3. In the Rhind Papyrus, Ahmes used a dot over a number to indicate a unit fraction. The dot derived from the hieroglyph for an open mouth-indicating that fractions were used to divide rations of food and drink.
4. Writing fractions stacked-using three terraces of type-with a horizontal line between the numbers goes back to Arabic writings. In *Liber Abbaci*, Fibonacci used stacked fractions.

The writing of the diagonal line for fractions, such as 2/3, first appears in a text published in Mexico in 1784, with a tilted flourish that slowly evolved into the less ornate /.

C. The symbols for exponents and powers come from Descartes.

1. Descartes developed the shorthand n^2 for n x a and n3 for n x n x n.

2. This notation allows us to write very large values in a compact fashion (e.g., 1,000,000.000 = 109).

D. Symbols for natural-number exponents also derive from Descartes.

1. In view of Descartes' notation, we have the law of exponents:

2. $\left(n^a\right)\times\left(n^b\right)=n^{a+b}$. We can combine products, sums, and positive powers to create polynomials; that is, such expressions as:

$3n^2 - 2n + 10$ or $120n^{201}+12.34n^{35}-111n^7-5n+27$

3. Polynomials offer an important way to classify numbers in a more refined manner than simply rational or irrational.

IV. The notation for rational exponents goes back to the 14th century.

A. What power of 9 equals $\sqrt{9}$ (We note that $\sqrt{9}$ = 3 because 3^2 = 9).

1. Suppose $9^p = \sqrt{9}$. Then we know that

$\left(9^p\right)\times\left(9^p\right)=\left(\sqrt{9}\right)\times\left(\sqrt{9}\right)=9$

2. By the law of exponents, we conclude that $9^{2P} = 9$

3. Thus, $2p = 1$ and $p = \frac{1}{2}$

4. We discover that that $9^{1/2} = 3$. Similarly, $2^{1/2} = \sqrt{2}$

B. Rational exponents signify roots: $a^{\frac{1}{2}} = \sqrt[b]{a}$ (the bth root of a).

Thus, $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ (because $2^3 = 8$). More generally.

$a^{\frac{c}{b}} = \left(\sqrt[b]{a}\right)^c$; thus, we can deduce a clear meaning for

rational exponents—for example, $8^{2/3} = \left(\sqrt[3]{8}\right)^2$ $(2)^2 = 4$.

C. Nicholas Oresme, one of the most influential philosophers and mathematicians of the 14th century, was the first to write about and make sense of rational exponents.

D. The convention of writing rational exponents became popular after Isaac Newton used the notation in a letter from 1676.

V. Irrational exponents also date back to the 14th century.

A. Nicholas Oresme was the first to consider such objects.

B. Recall that the rational numbers are dense on the number line; that is we can find rational numbers as close as we wish to any given real number. We use $2^{\sqrt{2}}$ as an example.

1. We can approach $4^{-}\sqrt{2}$ (which equals 1.414213562373095048...) using the rational numbers:

$\frac{14}{10}, \frac{141}{100}, \frac{1{,}414}{1{,}000}, \frac{14{,}142}{10{,}000}$ and so forth.

2. We can approach the value of $2^{\sqrt{2}}$ - by computing $2^{\frac{a}{b}}$ .in which $\frac{a}{b}$ is a rational number approaching . $\sqrt{2}$

3. We compute:

$2^{1.4}$ =2.6390158215457885187480039 42...

$2^{1.41}$ = 2.6573716281930231619683032 25...

$2^{1.414}$ = 2.6647496501840435422805296 59...

$2^{1.4142}$ = 2.6651190885323514691565805 65...

…

$2^{1.41421356237950}$ = 2.6651441426902250984971247 70...

$2^{1.41421356237309504}$ = 2.6651441426902251723906107 01...

and see that these values converge on a particular number.

4. We define $2^{\sqrt{2}}$ to be the number we are approaching in this manner. Specifically. $2^{\sqrt{2}}$ s the real number whose decimal expansion begins 2.6651441426902....

VI. Sometimes, our ever-widening vision of number is further enhanced by our understanding of arithmetic and by the machinery of mathematics.

Questions to Consider:

1. Which of the following expressions are polynomials?

$$5x^3 - 6x + 17, \ \sqrt{x^2+1}, \ \frac{3}{2}x^5 - 0.6x^2 + x - \frac{27}{13}, \ \frac{3x^2-8}{x=1}$$

Simplify $25^{3/2}$ and $8^{2/3}$ as much as possible.

2. The use of variables, exponential notation, and other mathematical shorthand allows for compact expressions and easier manipulation of complicated formulas. To what extent is this shorthand a benefit or a hindrance to learning, working with, and understanding mathematics?

Lecture Thirteen
The Story of π

Scope: In this lecture, we will tell the story of one of the most famous numbers in human history and one of the most important numbers in our universe. Although the origins of π (pi) are not known for certain, we do know that the Babylonians approximated π in base 60 around 1800 B.C.E. The constant it helps us understand our universe with greater clarity. In fact, the definition of π inspired the notion of a new unit to measure angles. This important angle measure is known as radian measure and gave rise to many insights into our physical world. As for π itself, Johann Lambert showed in 1761 that it is an irrational number. Ferdinand von Lindemann proved in 1882 that π is not a solution to any polynomial equation with integer coefficients. We will also see that many questions about π remain unanswered. We will close this lecture with a number of entertaining stories involving π .

Outline

I. We begin with some basics.

A. An interesting observation involving circles brings us to π .

1. Suppose we compare the length of the circumference of a circle to its diameter.

2. We discover that the circumference is slightly greater than three times the diameter.

B. The number it is defined to equal the ratio of the circumference of any circle to its diameter.

1. This ratio is constant no matter the size of the circle.

2. The first 30 digits in the decimal expansion air are: 3.14159265358979323846264338327 9.

1. We use the Greek letter π (pi) for this constant because the Greek word for periphery (the precursor to perimeter and circumference) begins with the Greek letter π .

2. The symbol π first appeared in William Jones's 1709 text A New *Introduction to Mathematics.*

3. The symbol was made popular by the great 18th-century Swiss

mathematician Leonhard Euler around 1737.

II. Attempts to produce the value of π began early in history.

A. The Babylonians approximated it in base 60 around 1800 B.C.E.

$$\pi = \frac{25}{8} = 3.125$$

B. The ancient Egyptian scribe Ahmes offered the approximation

$$\frac{256}{81} = 3.160493827...$$

C. An implicit value of π is even given in the Bible: In 1 Kings 7:23, a round basin is said to have a 30-cubit circumference and a 10-cubit diameter, implying that π = 3.

D. In 263 C.E., the Chinese mathematician Liu Hui believed that = 3.141014 .

E. Approximately 200 years later, the Indian mathematician and astronomer Aryabhata approximated π with the rational

$$\frac{62{,}832}{20{,}000} = 3.1416$$

F. Around 1400, the Persian astronomer Ghyath ad-din Jamshid Kashani computed π correctly to 16 digits.

III. The number π is everywhere.

A. The number π is used as a measure.

1. Given its connection with circles, we are led to a measure of angle as distance.

2. With the angle measure of degrees, we recall that one complete rotation has a measure of 360 degrees—the approximate number of days in one complete year.

3. We consider a circle with radius 1. Traveling around the circle once would produce a circumference of 2π ; thus, every angle corresponds to a distance measure part way (or all the way) around the circle.

4. We call this measure of angles radian measure; thus,

180 degrees = π radians, and 90 degrees = _$^{\pi}$ radians.

5. Radian measure is a much more useful measure of angles for

mathematics (including calculus and physics).

6. The term "radian" first appeared in print in the 1870s. but mathematicians, including the great Leonhard Euler, had been measuring angles this way for more than 100 years.

B. The number π appears in countless important formulas and theories, including the Heisenberg uncertainty principle and Einstein's field equation from general relativity.

C. Using π , we can calculate the area of a circle.

1. Given a circle having radius r, we know its circumference equals 2π r.

2. Now suppose we cut up the circle into tiny, pizza-like slices.

3. If we reposition the slices in an alternating up-and-down fashion, we can approximate the area of the circle by computing the area of the rectangle-like object we have created.

4. We discover the important formula that the area of a circle of radius *r* equals pai^{r}, a formula proven by Archimedes.

IV. The 18th-century German mathematician Johann Lambert showed that π is an irrational number in 1761.

A. We now know that the decimal expansion for π will never become periodic.

B. Given that π is not equal to any number of the form $\frac{m}{n}$ for

integers *m* and *n*, we see that nπ - *m* will never equal 0. Therefore, π will never be a solution to a linear polynomial equation *nx* - *m* = 0 with nonzero integer coefficients *m* and *n*.

C. The German mathematician Ferdinand von Lindemann generalized Lambert's work in 1882 and proved that π is not a solution to any polynomial equation with integer coefficients.

V. Some questions remain open to this day about it.

A. Are the digits of π random: that is, is π a normal number?

B. Is there any pattern in the digits of π ?

C. Is 2^{π} an irrational number?

VI. The number π has been involved in some amusing antics.

A. Around 1600. German mathematician Ludolph van Ceulen used a method of Archimedes's to compute the first 35 digits of π . He was so proud of his calculation that he requested those 35 digits be carved into his gravestone; his request was honored at the time of his death in 1610.

B. In 1873. British math enthusiast William Shanks computed the first 707 digits of π . It took Shanks more than 20 years to perform the necessary computations. (In 1944, D. F. Ferguson found that Shanks had made a mistake.)

C. In 1897, the Indiana General Assembly passed a bill declaring that π was equal to 3.2 (the state senate postponed the bill indefinitely).

D. Today, the first trillion digits of π are known. and billions of digits can be generated on a laptop computer with the appropriate software.

E. If we use the first 10 decimal digits of n to compute the circumference of the Earth's equator (assuming we know the exact diameter), we will be off by less than 0.2 mm.

F. The current world record holder for naming digits oft is Akira Haraguchi. who correctly recited the first 100,000 digits on October 3, 2006.

G. In order to remember the first 15 digits of π . remember the following (consider the letter counts in each word): *How I need a drink, alcoholic of course, alter the heavy lectures involving ancient constants.*

Questions to Consider:

1. Convert the following angles from degrees to radians: 45°, 60°. 30°.

2. Why do you think number enthusiasts keep uncovering more and more digits in the decimal expansion of it?

Lecture Fourteen
The Story of Euler's e

Scope: Compared with the number π . *e* is new to the number theory scene, but its popularity can only be described as exponential. First approximated by Jacob Bernoulli and first referenced by John Napier in 1614. this number quickly became one of the most important numbers in mathematics. Leonhard Euler in 1727 was the first to name it *e;* today, the number is known as "Euler's number." With a value of approximately 2.71828182845. this special number is fundamental in our understanding of growth. Armed with the official definition of *e.* we will consider several of its famous features, including writing *e* as an infinitely long sum of fractions that includes the mathematical exclamation !—an important representation that allowed Joseph Fourier, in 1815, to devise a clever proof that *e* is irrational. Although the complete decimal expansion for *e* remains unattainable, there is an attractive way to write *e* as an infinitely nested faction. In 1873. Charles Hermite extended Fourier's work and showed that *e,* just like π , is not the solution to any polynomial equation with integer coefficients. The number *e* is very important in calculus and. thus, has captured the admiration of science fans all over the world.

Outline

I. At the very end of the 17th century. Swiss mathematician Jacob Bernoulli was working through a computation involving compound interest, and he came upon an interesting number.

A. Suppose that we invest $1.00 into a savings account paying 100% interest per year. How much money would we have at the end of the year?

1. The answer depends on how often the interest is compounded. If the interest is paid once a year. then we end the year with $2.00.

2. *If* the interest is awarded twice a year, then after the first six months, we would have $1.50 (the interest of $0.50 equals half of 100% of $1.00), and after the second six months, we would earn half of 100% of $1.50, which equals $0.75; we end the year with S2.25.

3. If the interest were compounded quarterly, then we end the year with $2.4414....

4. If the interest were compounded monthly, then we end the year with $2.61303....

5. If the interest were compounded weekly, then we end the year with $2.6925....

6. If the interest were compounded daily, then we end the year with $2.71456....

7. If the interest were compound continuously, then we end the year with $2.718281....

B. We see that there is a limiting amount to our investment as the compounding becomes more and more frequent.

C. Bernoulli was the first to realize that this process has a limiting value and was the first to compute this special number, which begins: 2.71828182845904523536....

II. The number known as Euler's *e* developed during the course of a little more than a century.

A. This special number was first referenced in 1614 in a book by the Scottish mathematician John Napier.

1. In *Miraculous Canon of Logarithms,* Napier describes the theory of logarithms—an idea that he invented.

2. A logarithm is a means of studying exponents.

3. Also in this important book are calculations Napier made that took him 20 years (in which he introduces this special number 2.71828...).

B. The name was first given to this number by Leonhard Euler in 1727, and *e* is often referred to as Euler's number.

C. Euler's constant is an extremely important number.

D. It appears in essentially all questions of growth and decay.

1. These are commonly referred to as "exponential growth" and "exponential decay."

2. Often, populations grow at a rate proportional to the size of the population. Modeling this type of growth involves the number *e*.

3. Computations of the decay of radioactive substances (half-lives) often involve the number *e*.

4. The number *a* is one of the most important numbers in the study of calculus.

III. There are many ways to explicitly and precisely define *a*.

A. Bernoulli's observation about the limiting value found by compound interest can be stated as the limiting value of $\left(1+\frac{1}{n}\right)^n$ as the number *n* (the number of times of compounding) approaches infinity.

B. Another important definition of *e* arises out of calculus. The number *e* can be expressed as an infinite sum of ever-shrinking rational numbers:

$$e = 1+\frac{1}{1}+\frac{1}{(2x1)}+\frac{1}{(3x2x1)}+\frac{1}{(4x3x2x1)}+\ldots$$

C. The product of all natural numbers up to a number *n* is called "n factorial" and is denoted as *n! (e.g.,* 5! = 5 x 4 x 3 x2 x 1= 120).

1. Given this short-hand notation, the previous sum that equals *e* can be written as:

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots$$

2. This infinite sum can be extended to hold for any power of e.

This formula is: $e^x=1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots$

D. Another formula involving *e* is as a continued fraction:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots}}}}}}}}}}$$

IV. Even though this number has been known to us for only about 400 years, it has generated a tremendous amount of interest.

A. In 1815, Joseph Fourier used the infinite sum description of *e* to prove that *e* is an irrational number; we conclude from our earlier discoveries that the decimal expansion for e is unending and will never become periodic.

B. Like π , because e is not equal to any number of the form $\frac{m}{n}$ for integers *m* and *n,* we will see that $n- m$ can never equal 0; thus, e will never be a solution to a (nontrivial) linear polynomial equation $nx - m = 0$ with integer coefficients m *and* n.

C. In 1873, Charles Hermite generalized this result and showed that e (just as with π) is not a solution to any polynomial equation with integer coefficients.

D. Many mysteries remain.

1. Is e a normal number?

2. Are the numbers $e + \pi$, $e\pi$. and π^e irrational?

Questions to Consider:

1. What do you believe it says about our universe that there exists a constant, *e,* that is abstract and unknowable yet intrinsic to fundamental processes of growth and life?

2. Which number do you believe is more fundamental in our universe: π or *e?* Is it possible that these two extremely important numbers are connected in some natural way?

Lecture Fifteen
Transcendental Numbers

Scope: Using π and *e* as two famous examples, we will introduce the general idea of transcendental numbers. We will see how numbers are either a solution to certain special equations (polynomial equations with integer coefficients) or not a solution to any such equation. If a number is a solution to such an equation, we call it an algebraic number; otherwise, we call it a transcendental number. Although it took thousands of years to show that transcendental numbers existed, it took Georg Cantor only 50 years after Liouville's seminal work to prove that -almost all" numbers are transcendental. Even though we know that real numbers teem with transcendentals, it remains a very difficult task to produce specific ones. David Hilbert, in his famous address at the 1900 Congress of Mathematics in which he listed 23 fundamental open questions of mathematics. challenged the mathematical community to prove the transcendence of $2^{\sqrt{2}}$. Although many mathematicians (including Hilbert himself) considered this question out of reach, Aleksandr Gelfond and Theodor Schneider independently gave an affirmative answer in the mid-1930s. We close our discussion of transcendental numbers by describing some questions that remain unanswered to this day.

Outline

I. Algebraic numbers are solutions to certain polynomial equations.

A. We identify such equations as polynomials with integer coefficients.

1. One of the most common practices in mathematics is the solving of equations.

2. The simplest equations we can imagine are of the form: $x - 2 = 0$ or $2x - 10 = 0$. These are examples of linear equations: $nx - m = 0$. If *m* and n are integers, then the solution $x = \frac{m}{n}$ =is a rational number.

3. Linear equations are special cases of the simplest class of equations we can have, known as polynomial equations—sums and differences of numbers multiplied by x and raised

to various natural-number exponents, all set equal to 0.

4. Examples of polynomial equations are: $x^2 - 2 = 0$.

   $5x^3 + 9x^2 - 35x - 7 = 0$, and $- -x^{25} + 49x^{17} + x^{10} - 12x = 0$

5. A value of x that makes the equation valid is a *solution* to the equation.

6. For this lecture, we will consider only polynomials having integer coefficients.

B. Solutions to polynomials having integer coefficients can be found in special cases through special formulas.

1. For quadratic polynomial equations (equations in which the highest power of x is 2), we have the quadratic formula to solve them; in general, it has been proven impossible to have a formula that will find the solutions to an arbitrary polynomial equation.

2. The desire to solve these polynomial equations goes back to the Babylonians in 2000 B.C.E., who studied the solutions to $x^2 - 2 = 0$. One solution to this equation is $x = \sqrt{2}$ as we have already noted, the Babylonians approximated this value in base 60.

3. Numbers that are solutions to such polynomial equations with integer coefficients are known as algebraic numbers because they are numbers that arise from using algebra to solve the equations.

4. Integers are all algebraic numbers. rational numbers are algebraic numbers, and some irrational numbers are algebraic (e.g., : $\sqrt[3]{5}$ is algebraic because it is a solution to the polynomial equation $x^3 - 5 = 0$).

5. Although algebraic numbers have no specific "look" to them. they often involve roots (e.g., $\sqrt{2}, \sqrt[3]{2}, \sqrt{5-7}, \sqrt{\frac{6+\sqrt[4]{22}}{9}}$.

II. How are transcendental numbers defined?

A. At one time, one of the most important questions in mathematics was: Are all real numbers algebraic numbers?

1. A number that is *not* an algebraic number is called a transcendental number, which led to the question of whether

or not transcendental numbers existed.

2. Gottfried Leibniz was probably the first to use the term *transcendenta,*. but the modern definition of a transcendental number probably did not arise until the next century in the work of Euler.

B. In 1844, Joseph Liouville produced a particular real number and proved that it could not be an algebraic number; that is, it could not be a solution to any polynomial equation with integer coefficients. His number, now known as Liouville's number, is the first number known to be transcendental.

1. Its value is: 1.110001000000000000000001000000000...0001000.... with a decimal expansion of only 0s and 1s (and with the number of 0s between the 1s growing at the rate of n!).

2. His number is irrational because its decimal expansion does not become periodic.

3. This result was one of the greatest triumphs in our understanding of number and has since led to some beautiful and important mathematics that continue to be generated by research mathematicians.

C. Our understanding of transcendental numbers is relatively limited. Establishing the transcendence of a particular number remains a challenging undertaking.

1. Given this reality, Georg Cantor, in the late 1880s. surprised the mathematics community when he showed that "almost all" numbers are transcendental, extending the previous observation that "almost all" numbers arc irrational.

2. In 1937, mathematical historian E. T. Bell described this vision in a very romantic manner, comparing the algebraic numbers to stars against a black sky.

3. Once again, we see the theme that the familiar numbers (algebraics) are, in fact, rare, while the more mysterious transcendental numbers are more commonplace.

III. David Hilbert's seventh question challenged a generation of mathematicians.

A. In 1900, the Congress of Mathematics, held in Paris, invited David Hilbert, one of the greatest mathematicians of the modern age, to

deliver an address reflecting on the then-current state of mathematics.

1. He posed 23 open questions that he believed would shape the future of mathematics.
2. Only 10 of his 23 questions have been solved completely.

B. The seventh question perplexed mathematicians for more than three decades.

1. This question involves a number that we studied earlier: $2^{\sqrt{2}}$.
2. Hilbert asked the following: Let *a* be an algebraic number not equal to 0 or 1, and let *b* be an irrational algebraic number. Is a^b a transcendental number?

C. Aleksandr Gelfond in 1934 and Theodor Schneider in 1935 proved that the answer is yes: those numbers are always transcendental.

1. The result is now known as the Gelfond-Schneider Theorem.
2. As a consequence of this powerful theorem, we will discover that $2^{\sqrt{2}}$ is a transcendental number.

IV. Open questions remain.

A. Are any of the following numbers transcendental $\pi + e, \pi^e, \pi^{\sqrt{2}}, e^e, \pi^\pi, 2^\pi, 2^e$?

B. We conjecture that all of them are, yet we cannot prove they are irrational.

Questions to Consider:

1. Show that $\sqrt[3]{2}$ is algebraic by producing a polynomial equation with integer coefficients for which $\sqrt[3]{2}$ .. is a solution.
2. Why is it so challenging to produce explicit examples of transcendental numbers?

Lecture Sixteen
An Algebraic Approach to Numbers

Scope: We will open with a historical look back at algebra—what it means, its symbols. and how it can distract lovers. Inspired by the study of algebra—in particular, solving polynomial equations—we will offer a mathematical story that opens with the natural numbers and naturally leads us to discover integers. rational numbers, and algebraic irrational numbers, such as $\sqrt{2}$. This story brings us to a seemingly innocuous or perhaps even ridiculous question: Is every algebraic number a real number? We will discover that the answer is no—there are equations (such as $x^2 + 1 = 0$) that have *no* solutions among real numbers. Girolamo Cardano. a 16th-century mathematician, was the first to consider square roots of negative numbers, calling them "fictitious" and "meaningless." This discovery led to the need to expand the notion of number and, in turn, to the development of complex numbers. Given this new, expanded view of number, we will ask if every algebraic number is a complex number. In terms of this algebraic approach to numbers, we will see, through the work of Carl Gauss, that when we ascend to the complex numbers, we have arrived at the ultimate notion of number. Although this observation is true using this algebraic lens, we will soon discover there are other mathematical lenses through which we can gaze into the world of number—and those visions lead US to different conclusions.

Outline

I. What is algebra?

A. At the heart of algebra, we find two numerical expressions that are equal to one another. Because some number (or numbers) in one of the expressions is *unknown*, the main mission is to figure out the number that represents that unknown quantity.

1. For example, $2 \times 5 = 10$ is a true equation, but nothing is hidden from sight.
2. The equation $2 \times x = 10$. however. has a mysterious quantity, *x.* Our mission is to find the number x that makes this equation true; that is, we must find a number with the property that. when multiplied by 2, the product equals 10 (in this case, x = 5).

B. Word problems have challenged people for thousands of years.

1. The book *Greek Anthology* from the 4th century B.C.E. contained many algebraic riddles.

2. For example, six people are to divide a heap of apples. The first person receives $\frac{1}{3}$ of the apples; the second $\frac{1}{8}$; the third. $\frac{1}{4}$ and the fourth $\frac{1}{5}$. The fifth person receives 10 apples, and the last person is given only 1. How many apples were there in the original heap?

C. Today, we often refer to the unknown as x.

1. In the Rhind Papyrus, Ahmes used a symbol for "heaps" for unknown quantities.

2. The ancient Chinese indicated unknowns by their physical position in the equation, while in Hindu works we see the unknown as a dot.

3. Brahmagupta and. later, Bhaskara used the names of colors to designate different unknown quantities, while in Arabic works from 900 C.E., different coins were used to represent unknowns and one unknown was referred to as "thing."

4. In 1637, Rene Descartes adopted the italic letters x, y. and for unknowns, the notation that is used today.

D. The always-present object in an equation is the equal sign, =. Robert Recorde. who published an algebra text in 1557, introduced the symbol =======for equality. This symbol was later shrunk down to the more modest =.

E. Algebra can be viewed as a pastime (e.g., the challenges of Bhaskara in *The Gem of Mathematics).*

II. We see the need to expand our net of numbers through the desire to find solutions to simple equations.

A. We begin with natural numbers.

1. If we consider linear equations. such as $x+2=5$, we see that these equations have solutions.

2. We run into trouble, however, if we consider $x+7=5$, because there is no number in our pretend universe that

satisfies this equation.

B. We now expand our notion of numbers so that these equations have solutions; therefore, we allow negative natural numbers to be considered as numbers.

C. If we consider the entire world of number to be composed solely of the collection of integers, then we face further troubles when we consider such simple equations as $3x=5$.

1. There is no integer that satisfies this equation.

2. Therefore. we must allow fractional numbers to be considered as numbers.

D. If we consider the entire world of number to be composed solely of the collection of rational numbers. then we face further troubles when we consider such simple equations as $x^2=2$.

1. There is no rational number that satisfies this equation.

2. Therefore, we must allow irrational numbers to be considered as numbers. We call these irrational numbers "algebraic numbers."

III. Cardano's work in this area was groundbreaking.

A. One of Girolamo Cardano's greatest works was a text he wrote in 1545 entitled *The Great Art.* in which he included a systematic method to solve cubic equations of the general form $x^3+ax=b$

B. Cardano also posed the following question: Divide 10 into two parts whose product is 40. The answers are $5+\sqrt{-15}$, $5-\sqrt{-15}$*;* however, what does $\sqrt{-15}$ mean?

C. To explore this question, we consider a simpler equation: $x^2+1=0$.

1. If there is a number x such that it is a solution to $x^2+1=0$, then $x^2=-1$,

2. In this case, x cannot equal 0 because $0^2 = 0$ (not -1); therefore. the number x must be a nonzero number.

3. Given that the square of any nonzero real number equals a positive number, there is no real number that satisfies this equation.

IV. For the first time in this course, we find a natural need to expand our notion of number beyond the real numbers of our number line.

A. We call these new numbers complex numbers. and we denote the special number $\sqrt{-1}$ as *i*.

B. We call the number *i* an imaginary number.

1. The name *i* was given by Leonhard Euler two centuries after Cardano's original work.
2. Because *i* is not real, it cannot be a rational number; hence, it is irrational.
3. The number *i* is an algebraic number because it is a solution to the polynomial equation $x^2 + 1 = 0$.

V. Armed with the special number $i = \sqrt{-1}$, we can define all the complex numbers as numbers that can be expressed as x + yi, for real numbers x and *y (e.g.,* 2 + 3i, $-\sqrt{3} + \pi$ i, and –3i).

A. We can also combine complex numbers.

1. By definition, i^2 equals -1.
2. We see that $i^3 = -i$ and $i^5 = i$ is that is, we return to the number. This repeating pattern continues.

B. We can visualize complex numbers by considering a geometric object with an extra dimension. This idea leads us to the complex plane as a means of visualizing, complex numbers. The complex number x + *yi* is the point *(x, y)* on the Cartesian plane.

VI. Gauss's Ph.D. thesis was an algebraic end of the line for numbers.

A. We recall that our algebraic development of number evolved through our desire to have solutions to polynomial equations with integer coefficients.

B. We have seen that we can even solve such equations as $x^2 + 1 = 0$. although the solutions are no longer real numbers; we just added one new number, the imaginary number *i*, to build all the complex numbers.

C. It can be proven that just by including *i* (and all complex numbers *x+ yi),* we have captured all the algebraic numbers.

1. Every polynomial equation with integer coefficients has solutions of the form x *yi*, in which x and y are real

algebraic numbers.

2. In 1799, child mathematics prodigy Carl Friedrich Gauss. in his Ph.D. thesis, proved that every polynomial equation with complex coefficients has a solution of the form x + yi in which x and y are real numbers.
3. This insight shows us that there is no need to create any additional numbers. after we include the imaginary number to ensure that all polynomials have solutions.
4. This can be expressed by saying that the complex numbers are -algebraically closed."

Questions to Consider:

1. Do you believe it is appropriate to use the word *imaginary* to describe *i?* Place your response in the context of the history of numbers (recall that Diophantus called negative numbers "absurd").
2. Simplify the following numbers as much as possible: i^2, i^4, i^5, i^{25} and $i^{-1,000,000}$

Lecture 17
The Five Most Important Numbers

Scope: Before leaving the complex numbers, we will return to their geometrical universe: the complex plane. Here, we will discover that we can describe complex numbers x + *iy* in an alternative geometric manner using an angle and a length. Armed with this new way of writing complex numbers and an observation regarding sines and cosines, we are led to arguably the most beautiful identity in mathematics, an identity that, in one equality, connects the five most important numbers: 0, 1, 7, *e*, and *i*. This incredible result is known as "Euler's identity," named after the great 18th-century mathematician Leonhard Euler. Though Euler is credited with first stating this identity, it was known before him. This identity, together with the Gelfond-Schneider Theorem, quickly proves that e^m is, in fact, a transcendental number. We thus leave the complex plane and our algebraic adventure, holding in our minds one of the most beautiful objects in nature: an equality that brings together the most important constants of our universe.

Outline

I. The five most important numbers are 0, 1, π, e, and i.

A. The numbers 0 and 1 have special properties.

1. The number 1 is the multiplicative identity *(a* x 1 = *a).*

2. The number 0 is the additive identity *(a* + 0 = *a).*

B. The numbers π and e have enabled us to understand our universe in a deeper way.

C. The number i can be used to generate all the complex numbers.

A. There is one equation that connects the numbers 0, 1, π, e, and is $e^{\pi i} + 1 = 0$

II. With these numbers, we can reach another view of the complex plane of numbers.

A. Every complex number can be expressed in the form x + *vi*. We can visualize these numbers not on the (real) number line but as points on the complex plane.

B. We now look at the complex plane with a different eye and see that points in the complex plane can be expressed in terms of an angle and a length.

C. Every complex number corresponds to a right triangle in the complex plane.

III. We revisit some basic ideas from trigonometry involving right triangles and their angles.

D. The ratio of various side lengths of a right triangle produces various trigonometric functions.

1. The sine of one of the two non-right angles of a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse.

2. The cosine of such an angle is the ratio of the length of the adjacent side to the length of the hypotenuse.

E. Suppose we have a complex number x + yi with the property that its associated right triangle in the complex plane has a hypotenuse of length 1 and an angle of A radians.

3. The x value is equal to cos A, and the y value is equal to sin A.

4. Extending this principle to "degenerate" triangles, we see that –1 = cos 7 + i sin π . Note that this formula captures the spirit of Cantor's work in writing functions as sums of sines and cosines.

5. We now state an important result that follows from the ideasof calculus: $\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \ldots$, and $\cos A = A - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \ldots$!

IV. A sum for e^x can be formulated.

A. Given the previous infinite sums, we are reminded of the infinite sum for e^x.

B. Recall that $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \ldots$

C. We now evaluate this sum for x

1. Recall the powers of is $i2 = -1$. $i^3 = -i$, $i^4 = 1$ and so forth.
2. We can rewrite our infinite sum into two parts: a real part with no i's and an imaginary part with an *i*.
3. We notice that these two infinite sums equal trigonometric functions and conclude that $e^{i\pi} = \cos \pi + i \sin \pi$.

V. Revisiting the number $\cos \pi + i \sin \pi$ brings us back to our five numbers.

A. We now recall that $\cos \pi + i \sin \pi = -1$.

B. We put all these discoveries together and realize that $e^{\pi} = -1$; equivalently, we have shown that $e^{\pi} + 1 = 0$. Thus, the five most important numbers come together in one identity.

1. This identity was first recorded in its modern form by Leonhard Euler in the 18th century, though English mathematician Roger Cotes had proved the result 30 years earlier.
2. Euler was also the first to consider exponents with imaginary numbers in 1740, but in 1825 Gauss wrote, "The true metaphysics of $\sqrt{-1}$ remains elusive."

VI. We can apply our amazing discovery to prove that e^{π} is a transcendental number.

A. Let us assume for the moment that e^{π} is an algebraic number (recall that *i* is an algebraic, irrational number).

B. We conclude from the Gelfond-Schneider Theorem that the number $(e^{\pi})^{i}$ is a transcendental number; however, we have just shown that $(e^{\pi})^{i} = e^{\pi i} = -1$, and –1 is algebraic (not transcendental).

C. We arrive at a contradiction; therefore, our assumption that e^{π} is an algebraic number must be false. We have just proved that e^{π} is a transcendental number.

Questions to Consider:

1. Locate the following complex numbers as points on the complex plane: $1 + i$, $2 - 3i$, $5i$, and $-.05 - 2i$.

L17 - The Five Most Important Numbers

2. What is implied about mathematics, nature, truth, and the universe that the five most important numbers all come together in one elegant formula?

Lecture Eighteen
An Analytic Approach to Numbers

Scope: In this lecture, we will study the real numbers from an entirely different vantage point. We build up the real numbers from the rational numbers in an *analytical* manner—that is, we develop ideas that are inspired by one of the most fundamental applications of numbers: measuring how close two objects are by computing the distance between them. This approach—first considered by Augustin-Louis Cauchy in the early 1800s and later made precise by Karl Weierstrass, Georg Cantor, and Richard Dedekind—not only led to a deeper understanding of the real numbers but also allowed us to make many ideas from other areas of mathematics (including calculus) more exact and arguments more rigorous. This seminal work was the inspiration and starting point for a modern branch of mathematics now known as analysis. At its very core, finding the distance between two numbers requires us to find the absolute value of the difference of those two numbers. Using this simple idea, if we assume that the entire -universe of number" is the collection of rational numbers, then we quickly see that there are lists of numbers whose terms are getting closer and closer together, yet the value they are heading toward is *not* a number; the collection of rational numbers has, in some sense, holes. We can construct the collection of real numbers by expanding our notion of number by filling in these holes: this leads us to a continuous real number line. We close this discussion by investigating a new absolute value, the p-adic value.

Outline

I. By the 1800s, there was a great desire among mathematicians to set the notions of mathematics—especially the ideas of calculus—on more rigorous ground.

A. Intuitive and geometrical arguments became less attractive as a desire for precision grew.

B. This desire led to a serious interest in finding the precise meaning of "real number."

C. In 1817, the Czech mathematician Bernhard Bolzano saw this need for a precise definition of real numbers. Although he was not as influential as others to come, his work in this direction inspired

some to call him the -Father of Arithmetization.-

II. Mathematicians started with a world of rational numbers—which was completely understood and wanted to use these familiar numbers to define all real numbers.

A. In the 1820s, Augustin-Louis Cauchy developed a reasonable-sounding definition of real numbers as a limiting process using the rational numbers. Although his work was a major step forward, it lacked the rigor to satisfy some of his contemporaries.

B. Karl Weierstrass, now known by many to be the -Father of Modern Analysis,- declared all unending decimal expressions to be called *numbers.* This, for the first time, put the real numbers on solid footing.

C. Richard Dedekind and Georg Cantor (who was one of Weierstrass's Ph.D. students) were also working at making the real numbers precise.

1. In the late 1800s, they articulated an idea that we have been taking for granted: Points on a line can be placed in a one-to-one correspondence with real numbers. This is called the Cantor-Dedekind Axiom.

2. This idea can be made more precise and leads to a method of defining real numbers that is now known as "Dedekind cuts.-

III. A modern view of real numbers further expands our notion of number.

A. We find sequences of rational numbers for which the numbers get arbitrarily close to one another. Such sequences are called Cauchy sequences.

1. Sometimes these numbers head toward a rational number and sometimes they do not (e.g., 1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213...).

2. In the previous example, we see a "hole" in which a number should reside; however, that value is not rational.

B. If we extend our notion of number from just the rational numbers to the rational numbers together with these undefined holes, then we produce the real numbers and the continuum that we can view as a line.

1. This description represents our current precise view of real numbers.

2. We say we *complete* the rational numbers and doing so produces the real numbers.

C. The 19th-century German mathematician Leopold Kronecker strongly believed that all numbers should be derived from the natural numbers and zero. He thought that rational numbers, irrational numbers, and complex numbers were derived from false mathematical logic.

IV. In all the 19th -century attempts to give a precise definition of real numbers, we encounter the notion of distance.

A. Terms in a sequence of rational numbers get close together, implying a means of measuring closeness.

1. We measure closeness by using the absolute value, denoted The absolute value of a number is its distance from 0 (e.g.,

$|4|=|4|, \ |-5|=5$

2. To determine how close two rational numbers are, we take the absolute value of the difference between those two numbers.

3. For example, the distance between 4 and 7 is 3, and this fact corresponds to $3=|4-7|$.

4. The precise definition of the real numbers depends fundamentally on the notion of distance or, more specifically, on the absolute value.

B. An absolute value is any formula that observes certain rules.

1. The absolute value of 0 must equal 0, and the absolute value of any nonzero number must be positive.

2. The absolute value of a product of two numbers is equal to the product of their absolute values. For example, $|(4)\times(-3)|=12=12$ and $|4|\times|-3|=12$.

3. The shortest distance between two points is a straight line; put mathematically $|A+A|\leq|A|+|B|$ (this is called the triangle inequality). For example, $3=|5-2|\leq|5|+|-2|=5+2=7$.

V. There are other absolute values on the rational numbers that measure distance in an arithmetic sense. The p-adic absolute value is a new absolute value.

A. To illustrate this new type of absolute value, let us consider the special case in which the prime p equals 3.

B. We define the 3-adic absolute value (denoted as $|\ |_3$) first on the natural numbers as follows: Given a natural number N, the 3-adic absolute value of *N* equals the reciprocal of the highest multiple of 3 that divides evenly into *N*.

1. For example, $|12|_3=|2\times2\times3|_3=\frac{1}{3}$, $=-^{1}$, $|36|_3=|2\times2\times3\times3|_3=9$ and $|100|$, $=1$.

2. If the number *N* is not evenly divisible by 3, then its 3-adic absolute value is defined to be 3^0, which as mentioned earlier in this course, equals 1.

C. For any prime number *p*, we define the p-adic absolute value (denoted as $|\ |_p$) on the natural numbers as follows: Given a natural number *N*, the p-adic absolute value of *N* equals the reciprocal of the highest multiple of p that divides evenly into *N*.

1. Note that $|0|=0$; for negative numbers, we just ignore the negative sign.

2. We can find the p-adic absolute value of fractions (that is, of rational numbers) by simply dividing the p-adic absolute values of the numerator and the denominator. For example,

$$\left|\frac{8}{45}\right|_3=\frac{|8|_3}{|45|_3}=\frac{1}{\frac{1}{9}}=9$$

Questions to Consider:

1. Compute the following p-adic absolute values when $p=2$:

$$|4|_2, \ |12|_2, \ \left|\frac{1}{6}\right|_2, \ |0|_2, \ \left|\frac{24}{25}\right|_2.$$

2. Why were number theorists so obsessed with rigor and precision in defining the real numbers?

Lecture Nineteen
A New Breed of Numbers

Scope: Many of us today might find it difficult to empathize with the Pythagoreans' resistance in accepting the notion that the irrationals should be considered numbers. Yet in this lecture, we will face numbers that will be as counterintuitive to us today as the irrational numbers were to the Pythagoreans several thousand years ago. Using an entirely new arithmetic measure of distance involving prime numbers, called the p-adic absolute value, we can measure the distance between two rational numbers. We will see that, when measured with this new notion of distance, all triangles are isosceles. In addition, we will see that the rational numbers still have holes—missing values. We find and fill the p-adic holes with p-adic Cauchy sequences, but doing this forces us to extend our collection of p-adic numbers even more. This collection appears as a totally abstract consequence of arithmetic with no meaning in our physical world. We will briefly describe, however, how these unnatural numbers seem to offer new insights and help describe some delicate ideas in quantum physics.

Outline

I. We begin with arithmetic absolute value.

A. We recall the definition of the p-adic absolute value 1p on the integers, as follows: |0| = 0 ; if N is a natural number, then |N| equals the reciprocal of the largest multiple of p that appears when N is written as a product of prime numbers (e.g., $|90|_3 = \frac{1}{9}$).

1. We see that, because there are infinitely many prime numbers, there are infinitely many absolute values on the rational numbers.
2. These new absolute values were discovered by the German number theorist Kurt Hensel in 1897.
3. In the 1930s, Ukrainian mathematician Alexander Ostrowski proved that the only absolute values on the rational numbers are those that are equivalent to either the usual, familiar absolute value or one of these new p-adic absolute values.

B. Using this p-adic absolute value, we can measure the distance

between two rational numbers.

1. For example, if we fix $p = 3$, then the 3-adic distance between 5 and 2 is: $|5\text{-}2|_3 = |3|_3$ |, $= \frac{1}{3}$; thus, 3-adically, 5 and 2 are *3* relatively close to each other. The distance between 0 and $\frac{1}{6}$ however, is: $\left|0-\frac{1}{6}\right|_3 = \left|\frac{-1}{(2\times 3)}\right|_3 = 3$; thus, 3-adically, $\frac{1}{6}$ and 0 are relatively far apart.
2. The p-adic absolute value measures how many factors of *p* there are in a rational number $\frac{r}{s}$. The more factors of *p* in *r*, the smaller $\left|\frac{r}{s}\right|_p$ is; the more factors of *p* in *s*, the larger $\left|\frac{r}{s}\right|_p$ is.
3. Closeness in this p-adic context means that the difference of the two rational numbers has a high power of *p* in its numerator.

II. Measuring the lengths of sides of triangles having rational-number vertices will show how p-adic value can offer greater structure than standard absolute value.

A. We now consider the lengths of the sides of triangles formed by rational numbers (we will assume that we have not yet built the real numbers).

1. If we consider a triangle having vertices 5, -1, and 7, then the lengths of the sides are the distances between these three numbers: $|5-(-1)| = 6$, $|5 - 7| = 2$, and $|-1 - 7| = 8$.
2. Each side length has a different length; hence, this is an example of a scalene triangle.

B. We now attempt to find the 3-adic lengths of the sides of the previously described triangle.

1. The lengths, measured 3-adically, are $|5-(-1)|_3 = \frac{1}{3}$, | $|5-7|_3 = 1$, and $|-1-7|_3 = 1$.
2. In this case, we see that two sides have equal lengths; thus, this 3-adic triangle is isosceles.

C. More generally, any triangle formed by three rational numbers will be an isosceles triangle when the lengths of the sides are measured p-adically.

1. Consider the triangle having vertices 0, 3, and 12. If we measure the lengths of the sides 3-adically, we have: $|0-3|_3 = \frac{1}{3}$, $|0-12|_3 = \frac{1}{3}$, and $|3-12|_3 = \frac{1}{9}$. Again, we find that two sides have equal lengths.

2. The more arithmetic p-adic absolute value offers greater structure than the usual absolute value.

III. Finding and filling in the p-adic holes with p-adic Cauchy sequences is challenging but follows a definite logic.

A. Consider the infinite sum: $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + \ldots$. When measured 3-adically, this sum is heading toward a number—but which number?

1. If we call the number *N*, then we see that:

$N = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + \ldots$ and $3N = 3 + 3^2 + 3^{'} + 3^4 + 3^5 +$

2. Subtracting the two equations:

$$\begin{aligned} N &= 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + \ldots \\ 3N &= \phantom{1 +{}} 3 + 3^2 + 3^3 + 3^4 + 3^5 + \ldots \\ -2N &= 1 \end{aligned}$$

We conclude that $N = -\frac{1}{2}$; when we measure closeness with 2 the 3-adic absolute value, we see the very surprising fact that

$$-\frac{1}{2} = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + \ldots$$

3. By the same reasoning, we see that the terms in the infinite sum $3 + 3^2 + 3^4 + 3^8 + 3^{64} + 3^{64} + \ldots$, are getting smaller and smaller.

4. Moreover, the numbers $3, 3+3^2, 3+3^2+3^4, 3+3^2+3^4+3^8 \ldots$ are getting closer and closer to each other and form a 3-adic Cauchy sequence.

5. We would like this sequence of numbers to head toward a number; unfortunately, because the exponents are doubling, this sum is not equal to a rational number or even a complex number.

B. We must extend our notion of number so that all 3-adic Cauchy sequences converge.

1. We formally declare the sum $3 + 3^2 + 3^4 + 3^8 + 3^{64} + 3^{64} + \ldots$ to be a new irrational number.

2. This new irrational number is analogous to one of the irrational *real* numbers we found in Lecture Ten: 0.1010010001000010000001....

C. If we consider the rational number but measure closeness with a *p-adic* absolute value, then we again see there are holes.

1. When we filled in the holes using the usual absolute value, we extended our notion of number and discovered the irrational real numbers.

2. In this p-adic context, if we fill in the holes using the p-adic absolute value, then we extend our notion of number and discover *new* irrational numbers, such as $3 + 3^2 + 3^4 + 3^8 + 3^{64} + 3^{64} + \ldots$

D. This new collection of numbers is not the collection of real numbers and does not reside on a number line. It is known as the collection of p-adic numbers.

IV. It may seem difficult, despite this mathematical explanation, to embrace the idea that p-adic numbers are actually numbers.

A. Mathematicians have accepted this concept for more than a century.

B. The p-adic numbers were first studied by Kurt Hensel in the late 1800s.

C. They provide a purely abstract arithmetic notion of number that is extremely useful in studying integers, prime numbers, algebraic numbers, and even transcendental numbers.

D. It might appear as though these strange and foreign p-adic numbers have no utility in our everyday world or in any universe

beyond the abstract universe of number.

1. Recently, physicists have usedp-adic numbers to create new models of space-time and new ideas about string theory and quantum mechanics.
2. We see that these numbers might help us better understand the nature of our "real" universe.

Questions to Consider:

1. Consider the triangle having vertices 0, 5, and 25. Compute the lengths of the three sides of this triangle 5-adically and verify that the triangle is isosceles.
2. Give an example of a 5-adic number that is irrational. Describe how *p-adic* numbers in general challenge our intuition about the notion of number.

Lecture Twenty
The Notion of Transfinite Numbers

Scope: What comes after we have exhausted *all* numbers? We will distinguish two basic types of number: ordinal and cardinal. We will then wonder how to extend these ideas past all numbers into the realm of the infinite, which humans have contemplated for more than two millennia. Although other intellectuals made contributions to this area, it was the pioneering, and highly controversial work of Georg Cantor in the late 1800s that laid the foundation for our understanding of the infinite. Cantor's realization was that "counting" infinite quantities leads us nowhere; instead, he returned to our ancient ancestors and their first notion of number. We thus revisit the idea of a one-to-one correspondence between two collections. If the individual objects from two collections can be paired in a one-to-one fashion, we declare those two collections to have the same cardinality. To illustrate this ancient idea within this abstract context, we consider a number of scenarios in which we compare different infinite collections of numbers. They will also confirm one of our intuitively rock-solid notions: Infinity comes in just one-unending-size.

Outline

I. There are two uses of numbers: ordinal and cardinal.

A. We can place objects in a certain order and refer to them by their placement in the ordering as first, second, third, and so forth. These numbers ($1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}, 6^{th} \ldots$) are known as ordinal numbers.

B. A different use for numbers is as a means of enumerating; that is, counting how many elements (or members) there are in a collection.

1. These numbers (0, 1, 2, 3 ...) are known as cardinal numbers.
2. The number of elements in a collection is called the cardinality of the collection.

C. We can extend each of these notions of number to infinity. Mathematicians have devoted a great deal of study to what are called transfinite ordinal and transfinite cardinal numbers.

II. As early as the 5th century B.C.E., scholars contemplated infinity.

A. Around n50 B.C.E., Zeno considered paradoxes involving an infinite number of steps.

B. In the 4th century B.C.E., Aristotle concluded that infinity did and did not exist. He recognized that the counting numbers had no end but also believed that an infinite object could not exist in the real world because it would be boundless.

C. Sometime between the nth and 1St centuries B.C.E., the Gains studied mathematics extensively and believed there were different types of infinity.

D. In a work published in 1638. Galileo observed that the natural numbers (1, 2, 3, n ...) could be placed in a one-to-one correspondence with the perfect squares (1, 4, 9, 16 ...).

1. His one-to-one correspondence was: 1 $\Leftrightarrow$ I, 2 $\Leftrightarrow$ 4, 3 $\Leftrightarrow$ 9, n $\Leftrightarrow$ 16 , and so forth (here, the double arrow, $\Leftrightarrow$ indicates the one-to-one pairing of numbers from each collection).

2. Galileo observed that an infinite collection (the natural numbers) had been put in one-to-one correspondence with a proper subcollection of itself.

3. He thought this seeming paradox was one of the challenges provoked by infinity.

III. The notion of one-to-one pairings resurfaced two centuries later.

A. The 19th-century German mathematician Georg Cantor was the first to put the notion of infinity on a firm foundation.

1. He needed this precision for his work in function theory and number theory.

2. His revolutionary ideas were grounded in a very simple reality: We cannot use ordinary counting methods to understand infinity.

D. Suppose now that we cannot count to 5. How would we know that the number of fingers on our left hand equals the number of fingers on our right hand?

1. We can pair them up in a one-to-one fashion.

2. This one-to-one pairing—comparing quantities rather than counting them—was humankind's first attempt at grasping the

idea of quantities.

3. Cantor showed that this ancient basic idea is the key to unlocking the mysteries of infinity.

E. Returning to Galileo's observation about infinity, we say that a one-to-one correspondence between two collections is a way of pairing the elements of two collections so that every element from the first collection is paired with exactly one element of the second collection, and every element of the second collection is paired with exactly one element of the first.

1. We say that two collections have the same cardinality if there is a one-to-one correspondence between the two collections; that is. the collections are equally numerous.

2. It is easy to see if finite collections have the same cardinality; we simply count and see if the counts are equal.

3. For infinite collections, we need to describe a one-to-one correspondence to verify that the collections have the same cardinality.

IV. We move from studying individual numbers to studying the size of collections of numbers.

F. Let N denote the entire collection of natural numbers—that is, N = {1, 2, 3, 4, ... }—and let E denote the collection of all even natural numbers; that is, E= {2, 4, 6, 8, …}

1. Do these two collections have the same cardinality?

2. recall Euclid's accepted -common notion- from 2,000 years earlier: "The whole is greater than the part."

G. We compare natural numbers with integers.

1. Let *Z* denote the entire collection of integers; that is, *Z* = {...-4, -3, -2 ,-1, 0, 1, 2, 3, n...}.

2. Do the collections of natural numbers and the collection of integers have the same cardinality? The answer is perplexing on first inspection.

H. We compare natural numbers with rational numbers.

1. Let Q denote the entire collection of rational numbers; that is, Q = {every fraction}

2. Do the collections of natural numbers and the collection of rational numbers have the same cardinality? The answer surprised many people.

Questions to Consider:

1. How do you define infinity? What does infinity mean to you and what images does the word evoke in your mind?
2. Imagine a collection of objects. Suppose we remove some members from this collection to produce a second collection; is it possible that these two collections have the same cardinality? Does your answer change if you further assume that the initial collection is finite?

Lecture Twenty-One
Collections Too Infinite to Count

Scope: Here, we will offer one of the most counterintuitive and surprising facts about our world: Infinity, just as the numbers discussed earlier, comes in different sizes. This incredible discovery was first made by Georg Cantor in 1874. We will tell his story and his struggle to have his outrageous (but absolutely correct) mathematical ideas accepted by the mathematical community. His argument, at once simple and subtle, proves that the collection of real numbers is a greater infinity than the collection of natural numbers. Although the natural numbers can be listed in an orderly fashion and, in some sense, can be counted, we cannot count the real numbers; collections larger than the natural numbers are known as uncountable collections. We will then connect our discussion on the sizes of infinity to our earlier discussions on the likelihood that a real number, selected at random, will be irrational and transcendental. We will describe how both the collections of irrational numbers and transcendental numbers are so large that they are, in fact, uncountable. We also will connect our observations with our previous discussions on the Cantor set; although this is a sparse set, we will see that it is also a robust set in that it, too, is an uncountable collection. Once we open our mind to the reality that there are at least two different sizes of infinity, we cannot help but wonder if there is a third size of infinity.

Outline

I. Countable collections lead us to challenge our notion of infinity.

A. We say that the cardinality of a collection is countable if it has the same cardinality as the collection of natural numbers or if the collection contains only finitely many elements.

1. The collection of cards in a deck of playing cards is countable because the number of cards in the deck is a finite number: 52.
2. Each of the following collections of numbers is a countable collection: the collection of even numbers, the collection of integers, and the collection of rational numbers.

B. It seems obvious that any infinite collection should have the same cardinality as any other infinite collection, but is this sensible-sounding assertion mathematically correct?

II. Cantor's ingenious insight was to look at one-to-one correspondence in a new way.

I. Suppose that the collection of real numbers is countable.

1. We will just consider the interval of real numbers between 0 and 1.
2. We will assume that the cardinality of natural numbers is the same as the cardinality of all real numbers between 0 and 1.
3. We thus assume that there *does* exist a one-to-one correspondence between the collection of natural numbers and the collection of real numbers between 0 and 1.
4. What might such a one-to-one correspondence look like?

A. Cantor used the digits of real numbers appearing on our assumed correspondence to generate a real number.

1. He focused on the digits that lie along the "diagonal" of the column of real numbers.
2. He then created a real number that does *not* appear on our list.
3. Each digit of this new real number was selected so that it differed from the corresponding digit along the "diagonal" digits.
4. Cantor then argued that this new number is a real number between 0 and 1 that never appears in our assumed one-to-one correspondence.
5. Our assumption that a one-to-one correspondence exists between these two collections is, thus, false. We are forced to conclude that the collection of real numbers between 0 and 1 is a larger collection than the collection of natural numbers; there is no way to pair them all up.
6. For this reason, there can be no one-to-one correspondence between natural numbers and the entire collection of real numbers.
7. Cantor's method of proof is now known as Cantor diagonalization.

B. Cantor discovered that infinity comes in more than one size.

1. Informally, the infinity that represents the cardinality of real

numbers is larger than the infinity that represents the cardinality of natural numbers.

2. The real numbers are *not* countable.
3. We call collections that are not countable -uncountable."

III. There were numerous reactions to this startling reality from the mathematics community.

A. In 1831, Carl Friedrich Gauss protested vehemently against Cantor's position.

B. In 1906, the French mathematician Henri Poincare wrote that there was no actual infinity; he saw the Cantorians as being trapped by contradictions.

C. Leopold Kronecker was a powerful opponent of Cantor's work, which established the existence of infinitely many irrational numbers (in fact, uncountably many).

1. Kronecker did not view irrational numbers as natural.
2. Given that Kronecker was perhaps the most powerful mathematician in Germany during his lifetime, his opposition to Cantor had an enormous impact.

D. Carl Weierstrass, Bertrand Russell, and David Hilbert were impressed with Cantor's work and defended Cantor to his detractors.

E. Despite the support of these mathematicians, Cantor struggled throughout his career.

1. Cantor spent his entire career at the University of Halle, a less prestigious institution than he felt he deserved.
2. He desired a position at the University of Berlin, the premiere German institution, but the department chair was Kronecker.
3. He imagined Kronecker's distress were he to obtain a position in Berlin.
4. Cantor was extremely confident about the truth of his work on infinity.
5. The controversy inspired by his work took a toll.

IV. Bearing Cantor's work in mind, we must revisit irrational and transcendental numbers.

A. We start with the cardinalities of the collections of rational numbers and irrational numbers.

1. Recall that the collection of rational numbers is a countable collection.
2. Because real numbers are uncountable, we conclude that the collection of irrational numbers is uncountable.
3. This discovery reflects back to our previous discussions of irrationals; again, we see our recurring theme of familiar and exotic numbers.

B. By a modified diagonalization argument, we can establish the fact that the collection of all algebraic numbers is countable; thus, we discover that the collection of transcendental numbers is uncountable.

C. Recall that the Cantor set is the collection of real numbers between 0 and 1 whose base-3 expansions contain only Os and 2s.

1. In view of our previous discoveries from Lecture Eleven, we deduce that this collection of numbers is uncountable.
2. In this way, we discover that there must be transcendental numbers in the Cantor set.

L21 - Collections Too Infinite to Count

Questions to Consider:

1. How could mathematicians of the day be so opposed to Cantor's ideas leading to different sizes of infinity?
2. The list below attempts to show a pairing between the natural numbers and the real numbers. Use Cantor's idea of diagonalization to produce the first seven decimal digits of a number that will not be on the list.

1 ⇔ 0.2736n810...

2 ⇔ 0.01926573...

3 ⇔ 0.22937510...

4 ⇔ 0.61100029...

5 ⇔ 0.71099058...

6 ⇔ 0.2938n655...

7 ⇔ 0.56n78392...

Lecture Twenty-Two
In and Out—The Road to a Third Infinity

Scope: The goal of this lecture is to construct a collection whose cardinality is greater than the uncountable cardinality of the collection of real numbers. To this end, we will follow Cantor's own creative path yet again and consider the abstract but attractive idea of power sets. A power set associated with a collection is the totality of all possible subcollections of the original set. After shoring up our understanding of this new idea through several simple examples, we will return to Cantor's diagonalization argument, which established that the cardinality of real numbers is greater than the cardinality of natural numbers. We will apply this argument to prove that, given any collection, the cardinality of its associated power set is greater than the cardinality of the original collection. We then will apply this principle to show that there exists an infinite collection that has a greater cardinality than the collection of real numbers.

Outline

I. We begin with an examination of sets.

A. A set is an abstract collection of elements.

1. For example, the collection that contains the Marx family—Chico, Groucho, Harpo can be viewed as a set with three elements: {Chico, Groucho, Harpo}.

2. Another example is the collection of the two most popular condiments in this country, ketchup and mustard: {ketchup, mustard}.

3. The term set first appeared in 1851 in a work by the Italian mathematician Bernhard Bolzano entitled Paradoxes of the Infinite.

B. A subset of a set is any collection that comes from a given set.

1. For example, {Chico, Harpo} is a subset of {Chico, Groucho, Harpo}.

2. An empty set is a subset that contains no elements.

3. The entire set is the other extreme example of a subset (called the improper subset).

II. Given a set, we define the power set of the set to equal the set whose elements are precisely all the subsets of the original set.

A. Our set of condiments, C = {ketchup, mustard}, has the following four subsets: the empty set, {ketchup}, {mustard}, and {ketchup, mustard}.

1. These four sets are the elements of the power set of {ketchup, mustard}.

2. We call the power set P(C); that is, P(C) = {empty set, {ketchup}, {mustard}, {ketchup, mustard}}.

B. Let S = {Chico, Groucho, Harpo}.

1. The power set of this collection, denoted as P(S), is the collection of all subsets of S; that is, P(S) = {empty set, {Chico}, {Groucho}, {Harpo}, {Chico, Groucho}, {Chico, Harpo}, {Groucho, Harpo}, {Chico, Groucho, Harpo}}.

2. Notice that the order in which we list the elements does not matter.

C. We compare the cardinality of a set with its power set.

1. Returning to the previous examples, we notice that the cardinality of C is 2, the cardinality of P(C) is 4; the cardinality of S is 3, and the cardinality of P(S) is 8.

2. The original set is smaller than its power set. Does this observation always hold?

III. Cantor's Theorem helps us answer this question.

A. Cantor was able to apply his powerful insights into sets to prove that even if a set is infinite, its cardinality will always be smaller than the cardinality of its associated power set. This result is known as Cantor's Theorem.

B. Cantor used his diagonalization idea to prove his result about power sets.

1. We illustrate his argument with our example involving the three Marx brothers.

2. We assume that, contrary to what we wish to establish. the set S and its power set P(S) have the same cardinality.

3. Given this assumption, a one-to-one correspondence between the elements of the set S and the elements of the set P(S) must exist. For example:

Chico	⇔	{Chico,	Groucho,	Harpo}	
Groucho	⇔	{		Harpo}	
Harpo	⇔	{Chico,	Groucho,	}	

4. Cantor observed that any element in S is either in a subset or not; there are exactly two possibilities.

C. Cantor modified his diagonalization idea to create a new subset of S.

1. If the element in the left column was in the collection to which it is paired (the corresponding collection in the right column), then he did not include this element in his new subset.

2. If the element in the left column was not in the associated subset in the right column, then he did include that element in his new set.

3. In our example, his diagonalization method would produce the subset {Groucho, Harpo}. We know, by the way we produced it, that this subset does not appear in the right column.

4. We can generalize this idea to see that no one-to-one pairing between a set S and its power set P(S) can exist—we can always find a missing element of P(S) whenever we try to pair up the elements from the two collections.

IV. An infinity exists beyond the cardinality of the real numbers.

A. Recall Cantor's discovery that the cardinality of the set of natural numbers is smaller than the cardinality of the set of real numbers; that is, the infinity of the set of natural numbers is a smaller infinity than the infinity of the set of real numbers.

B. How can we find a collection whose cardinality is larger than the cardinality of the collection of real numbers?

1. We can consider the power set of real numbers; that is, we can consider the collection of all subsets of real numbers.

2. The power set of real numbers has a greater cardinality than the set of real numbers, which means there is a third size of infinity!

Questions to Consider:

1. Consider the collection of suits in a deck of cards: {♣, ♠, ♥, ♦}. How many elements are there in the power set of this collection? Can you list them all?

2. Do you believe there is a collection that has a greater cardinality than the cardinality of the power set of the real numbers, or do you think that the power set of the real numbers has the largest possible cardinality?

Lecture Twenty-Three
Infinity—What We Know and What We Don't

Scope: In this lecture, we will visit the frontiers of our understanding and discover what humankind knows and does not know about infinity. Are there infinitely many different sizes of infinity? Is there a largest infinity (the "mother of all infinities")? Is the cardinality of the collection of real numbers the next infinity after the cardinality of the collection of natural numbers, or is there an infinity between the two? We are able to answer two of these questions in view of our previous discoveries; the remaining question, however, has a shocking answer. This question. which preoccupied Cantor at the end of his life, remained unanswered for decades. Here, we will share the interesting 20th-century story of that question, known as the Continuum Hypothesis. As we will see, this question leads us to one of the cliffs of mathematics. Its unusual answer, found by Kurt Gödel and Paul Cohen in the mid-1900s, literally brings us to the very edge of that cliff. We will take a moment to enjoy that incredible sight, then carefully take one step back to avoid falling into the abyss.

Outline

I. Once our simplistic view of infinity collapses under the weight of Cantor's mathematical arguments, other previously unimaginable questions arise.

II. Are there infinitely many infinities?

A. In the previous lecture, we considered Cantor's argument that showed that the power set of any set has a greater cardinality than the original set.

B. By applying Cantor's result, we see the answer to this question is yes.

C. If we consider the power set of the power set of the collection of real numbers, then we have produced a collection having greater cardinality than the cardinality of the power set of the real numbers; thus, we have just produced a fourth size of infinity.

D. If we repeat this process, we discover that there are infinitely many different sizes of infinity.

III. Is there a largest infinity?

A. Another intriguing question is to wonder if there is one all-encompassing infinity.

B. The answer is no; if we had such an infinite collection, then we need only consider its power set to produce a collection that, by another application of Cantor's Theorem, must be larger.

IV. Is there an infinity between the sizes of the natural numbers and the real numbers?

A. We recall that the cardinality of the natural numbers is smaller than the cardinality of the real numbers.

B. This question leads to the Continuum Hypothesis, which states that the cardinality of the collection of real numbers—the cardinality of the continuum—is the next-to-largest infinity after the countable collection of the natural numbers.

1. Cantor was the first to pose the Continuum Hypothesis and worked very hard to prove it, but he was unable to resolve the issue.

2. David Hilbert listed the Continuum Hypothesis as the first challenge in his list of 23 challenges at the turn of the 20th century.

C. For Cantor, the Continuum Hypothesis was almost an obsession. This obsession, along with his combative relationship with Kronecker and others, had a serious and negative impact on his life.

1. Between 1874 and 1884, Cantor published his seminal papers on infinity, including six papers that formed the foundation of modern set theory. In 1884, he suffered his first major depression and was plagued by mental health problems for the remainder of his life.

2. During the last 30 years of his life, Cantor was in and out of mental clinics. His problems grew with the loss of his mother in 1896 and his youngest son in 1899.

3. Though he continued to do valuable research, his focus on mathematics declined. At one point, for example, Cantor

4. turned his energy to proving that Francis Bacon was the author of Shakespeare's plays.

5. After finally seeing the mathematical community acknowledge and celebrate his incredible contributions to mathematics, Cantor died of a heart attack in a sanatorium in

1918 at the age of 73.

V. The Continuum Hypothesis, a seemingly straightforward assertion, has a very strange resolution.

   A. The statement, in fact, resides outside the domain of mathematics; that is, it can be shown to be neither true nor false within the narrow confines of mathematics.

   B. The issue is extremely deep and involves advanced work in logic and set theory.

      1. In 1940. Kurt Gödel showed that the Continuum Hypothesis cannot be disproved using tools and theorems from mathematics.

      2. In 1963, Paul Cohen showed that it cannot be proved using mathematical machinery.

      3. We thus say that the statement is independent.

      4. Two independent launches into the galaxy of mathematics reveal that the world of mathematics would look the same if we assume that the Continuum Hypothesis is true or if we assume it is false.

   C. The study of number and the notion of comparing sizes of collections have brought us to one of the cliffs of mathematics itself.

Questions to Consider:

1. Suppose a friend said to you that she had an infinite collection of which she was very proud because it was so vast. How would you gently put her in her place by describing a collection that has greater cardinality than hers?

2. How have these lectures on infinity changed your view of this abstract mathematical notion? How have these discussions on different sizes of infinity challenged your beliefs and your view of the world?

Lecture Twenty-Four
The Endless Frontier of Number

Scope: In this last lecture, we will reflect back on our journey through numbers. We have seen that numbers are abstract objects of the mind that allow us to understand our world and our lives with greater clarity. For the number theorist and enthusiast, however, numbers are objects of independent beauty, intrigue, and curiosity. In this lecture, we open with a sense of what number is and how it evolved from a practical necessity to a creative art. We will consider the structure of numbers and explore the many means by which we can classify them to better understand and appreciate their individual nuances. We will then face the question "What is number?" and explore how that notion continues to evolve toward number theory as we understand it today. We will step back and consider the logic and rigor that underscored our entire exploration. The ever-present rigorous proof is both a science and an art. We will offer a window into the modern world of mathematical research and see how, in practice, those frontiers move outward. Finally, we acknowledge and celebrate the human passion involved in moving our notion of number forward. The quest to understand and tame the notion of number has transcended human history and cultural divides; what makes this intellectual quest so universally appealing? What surprising and intriguing plot twists lie ahead in the story of number? That enticing, unending journey will fuel the creativity and imagination of our descendants for the next 1,000 years.

Outline

I. A look back at our course takes us from the dawn of quantification through the struggle to communicate quantitatively, from number as an attribute to number as an object.

   A. Once we name abstract objects, they exist in our minds.

   B. Numbers have captured the imaginations of all people from all cultures throughout human history. It is an intellectual curiosity that brings humankind together.

II. We explored various perspectives, particularly an algebraic approach and an analytic one.

   A. An algebraic approach of solving polynomial equations allows us to

discover i, the complex numbers, and the distinction between algebraic and transcendental numbers.

B. An analytic approach of measuring distance allows us to give a precise definition of the numbers on a real number line and explore parallel universes of numbers, such as the p-adic numbers.

III. What is number?

A. We have come to see an ever-evolving notion of number as we have journeyed along the intellectual paths toward an understanding of the theory of numbers.

B. Every time we develop a new insight into numbers, it forces us to rethink our old notion of what number means.

C. Numbers hold many surprises—often what first appears strange in our minds is. in fact, ordinary.

D. Numbers have a power and import both within the universe of mathematical ideas and far beyond.

E. Are numbers discovered or created?

IV. Number theory today is something our early ancestors probably couldn't have imagined.

A. There are many different areas of number theory.

B. The two main branches of modern number theory are analytic and algebraic number theory.

1. Analytic number theory employs the ideas of calculus to answer questions about numbers.

2. Algebraic number theory focuses on the study of numbers that arise from solutions to polynomial equations with integer coefficients—the algebraic numbers.

C. Mathematics moves from observations to conjectures to rigorous proof.

D. Mathematicians at once balance rock-solid truth and wild, unbridled creativity.

E. Mathematicians must first stumble upon some structure or pattern, believe that this form is a general principle, then verify that this principle is valid through a clear, correct, and complete logical argument that establishes its validity.

V. A culture of number theory research exists, and within it, we find scholars who extend our boundaries of understanding, despite—and sometimes because of—their failures.

VI. The study of number is not a science devoid of human emotion. and throughout our course we have seen great passion as humankind struggled with the idea of number.

A. We have seen some very deep and abstract ideas within mathematics.

B. We also have experienced the joy and pleasures generated by discovery, creativity, and imagination.

C. We have witnessed for ourselves that within numbers we find beauty, elegance, grace, and mystery.

D. Many questions remain, and we can all appreciate and contribute to the quest to conquer new frontiers ahead.

Questions to Consider:

1. In Lecture One, you were invited to write a definition of number. How do you view your original answer now? Extra Challenge: Are numbers created or discovered?

2. As we have seen in this course, the notion of number evolves with humankind's intellectual development. What other great ideas have followed a similar evolutionary trajectory?

Timeline

30.000 B.C.E	Paleolithic peoples in Central Europe and France use notched bones as counting tools.
c. 4000 B.C.E	Sumerians use clay tokens (calculi) to represent quantities of different items.
c. 3500-3000 B.C.E	Sumerians record numeral symbols on clay tablets (an abstraction from the token system used previously); Babylonians develop base-60 numeral system; Egyptians use pictographs for numbers.
c. 3000 B.C.E	Sumerians develop cuneiform, a system of writing that includes distinctive numbers.
c. 2000 B.C.E	Babylonians and Egyptians use fractions.
c. 2000-1650 B.C.E	Babylonians apply the Pythagorean Theorem to approximate $\sqrt{2}$.
c. 1650 B.C.E	Rhind Papyrus is written (copied by the scribe Ahmes), showing extensive Egyptian calculation techniques, including an approximation to 7 of 3.16.
c. 1400 B.C.E	Chinese use base-10 numeral system.
1000 B.C.E	Chinese book gives the first record of a magic square.
c. 540 B.C.E	Pythagoras founds his school and proves the Pythagorean Theorem; the Pythagoreans and the Jains in India begin the earliest explorations of abstract properties of numbers; later,the Pythagoreans are confounded by the irrationality of $\sqrt{2}$.
c. 400 B.C.E	The Hindu numerals 1-9 begin to develop in India.
300 B.C.E.	Euclid presents his axiomatic method for geometry in Elements and proves the infinitude of primes and the irrationality of -N5 , while his common notions form the basis for modern arithmetic; Babylonians have a symbol for zero as a placeholder.

c. 100 B.C.E.	Chinese solve equations with negative numbers.
c. 250 C.E.	Mayans use a base-20 numeral system, including symbols for zero as a placeholder.
c. 650 C.E.	Brahmagupta understands zero as a number, not just a placeholder, and uses negative numbers systematically; Bhaskara 1 uses the symbol 0 for zero.
c. 800 C.E.	Arab mathematicians adapt and promote the use of Hindu numerals.
1202	Fibonacci brings knowledge of Islamic mathematics to Europe, encourages the adoption of the base-10 numeral system, and writes on the Fibonacci sequence.
1350	Oresme offers the first clear treatment of fractional exponents.
1489	The first appearance of + and – signs occurs in a German arithmetic book by Widman. 1545 Cardano introduces the idea of the square root of a negative number, leading to the discovery of complex numbers.
1585	Simon Stevin offers the first written reference to the number line and the first full account of the decimal expansion of numbers.
1614	Napier gives the first reference to the number e.
1632	Galileo discovers an apparent paradox concerning infinite sets.
1637	Descartes develops superscript (exponential) notation for powers of numbers and begins using x, v, and z to denote unknown quantities.
1713	Bernoulli approximates e using a

formula for compound interest.

1727 Euler is the first to name e.

1740 Euler is the first to consider imaginary numbers as exponents.

1761 Lambert shows that 7 is irrational.

1799 Gauss shows that every polynomial equation has its solutions within the complex numbers.

1815 Fourier shows that e is irrational.

1820 Cauchy defines the real numbers using infinite sequences of rational numbers and a limiting process.

1844 Liouville constructs the first example of a transcendental number.

1873 Cantor shows that the infinity of real numbers is larger than the infinity of natural numbers. Hermite shows that e is transcendental.

1877 Cantor proposes the Continuum Hypothesis.

1882 Lindemann shows that pi is transcendental.

1883 Cantor defines the set of real numbers between 0 and 1, known as the Cantor set.

1891 Cantor proves that the cardinality of a set is always smaller than the cardinality of its power set (now known as Cantor's Theorem).

1896 Hadamard and de la Vallee Poussin independently prove the prime number theorem.

1897 Hensel defines the p-adic absolute value.

1900 Hilbert poses 23 questions at the Second International Congress of Mathematics in

Paris as a challenge for the 20th century.

1902 Hensel defines the p-adic numbers.

1909 Borel introduces the concept of normal numbers and shows that the chance a random real number is normal in base 10 is 100%.

1940 Gödel establishes that the Continuum Hypothesis cannot be disproved within the standard axioms of mathematics.

1963 Cohen establishes that the Continuum Hypothesis cannot be proved within the standard axioms of mathematics.

2006 The 44th Mersenne prime is found. It equals $2^{232582657}-1$ and has 9,808,358 digits.

Glossary

abacus: A calculation device in which beads representing numbers are strung on wires, allowing for speedy arithmetic.

absolute value: The distance of a real number from zero on the number line.

additive identity: Zero is the additive identity because a + 0 = a for any number a.

additive inverse: The additive inverse of number a is -a because a + -a = 0, which is the additive identity. For example, the additive inverse of 5 is -5, and the additive inverse of -17 is - (-17) = 17.

additive system: A numeral system in which symbols represent specific values. A number represented by a collection of symbols equals the sum of the values of the individual symbols.

algebra: The branch of mathematics that studies equations, their solutions, and their underlying structures.

algebraic number theory: The branch of number theory that studies numbers that are solutions to certain polynomial equations.

algebraic numbers: The collection of all numbers that are solutions to nontrivial polynomials with integer coefficients.

algebraically closed: Complex numbers are algebraically closed because every polynomial equation with coefficients from the complex numbers has all its solutions within the complex numbers.

"almost all": A portion of a collection is said to be "almost all" of that collection if, when an item is selected at random from the entire collection, the chance of choosing something inside that portion is mathematically 100%.

amicable numbers: Two numbers are amicable if each is equal to the sum of the proper divisors of the other.

analysis: The branch of mathematics that generalizes the ideas from calculus, especially notions of distance and continuous change.

analytic number theory: The branch of number theory that studies integers (especially primes) using ideas from calculus and analysis.

axiom: A fundamental mathematical statement that is accepted as true without rigorous proof.

Babylonian: A dominant culture in Mesopotamia during much of the 2nd

millennium B.C.E. The Babylonians developed a true place-based numeralsystem in base 60 and approximated $\sqrt{2}$ to seven decimal places.

Barber's paradox: Suppose there is a town in which all the men shave and a lone barber shaves exactly those men who do not shave themselves. The question is: Who shaves the barber? If he does not shave himself, then he must shave himself but if he does shave himself, then he must not shave himself. This paradox is attributed to Bertrand Russell.

barred gate: A symbol for 5 consisting of four vertical slashes crossed with a single diagonal slash.

base-2 numeral system: A positional system using only the numerals 0 and 1, with the value of the digit equal to itself times a power of 2. For natural numbers, the rightmost position is the face value of the digit. Each position to the left has a higher power of 2. Also called the binary system.

base-3 numeral system: A positional system using only the numerals 0, 1, and 2, with the value of the digit equal to itself times a power of 3. For integers, the rightmost position is the face value of the digit. Each position to the left has a higher power of 3. Also called the ternary system.

base-10 numeral system: A positional system using the numerals 0, 1, 2 ... 9, with the value of the digit equal to itself times a power of 10. For integers, the rightmost position is the face value of the digit. Each position to the left has a higher power of 10. Also called the decimal system.

base-60 numeral system: A positional system using the numerals from 0 to 59, with the value of the digit equal to itself times a power of 60. For integers, the rightmost position is the face value of the digit. Each position to the left has a higher power of 60.

binary: See base-2 numeral system.

Botocudos: An indigenous tribe from what is now eastern Brazil. The Botocudos have a primitive knowledge of numbers; their language does not include names for numbers beyond 2.

calculi: The Greek word for "pebbles," calculi were pebbles or clay tokens used to represent numbers or for basic counting.

calculus: The branch of mathematics that studies continuous processes and instantaneous rates of change based on precise measures of distance.

Cantor-Dedekind Axiom: Points on a line can be placed in a one-to-one correspondence with real numbers.

Cantor set: The set of real numbers between 0 and 1 whose base-3

expansions contain only the digits 0 and 2.

Cantor's Theorem: The cardinality of the power set of a set is always larger than the cardinality of the set itself

cardinal number: A number that represents the size of a collection. Also called cardinality.

cardinality: The cardinality of a set is a quantity that represents the size of the collection. Also called the cardinal number.

Cauchy sequence: An infinite list of numbers that get arbitrarily close together.

coefficient: In a polynomial, a coefficient is the number multiplied by an unknown power (e.g., in the polynomial $27x^8+7x^3-8x$, 27 is the coefficient of x^8).

complex numbers: The collection of all numbers of the form x yi, where x and y can equal any real number and i is the square root of - 1. complex plane: A representation of the complex numbers consisting of a plane with horizontal (real) and vertical (imaginary) axes meeting at a right angle at a point called the origin.

Congress of Mathematics: One of the largest and most important conferences for mathematicians in the world, it has been held approximately every four years since 1897. The Congress of 1900 was marked by David Hilbert's announcement of 23 open questions, which included the Continuum Hypothesis.

conjecture: A mathematical statement thought to be true but for which a rigorous proof has not yet been found.

continuum: The collection of real numbers.

Continuum Hypothesis: Cantor's conjecture that there is no size of infinity .between the cardinality of the natural numbers and the cardinality of the real numbers.

cosine: The cosine of an angle of a right triangle is the quotient of the length of the side adjacent to the angle divided by the length of the hypotenuse.

countable: A set is countable if it is finite or there is a one-to-one correspondence between its elements and the natural numbers.

counting numbers: The collection of numbers 1, 2, 3, 4, 5, and so on. Also called the natural numbers.

cuneiform: One of the earliest forms of writing, invented around 3000 B.C.E. by the Sumerians.

decimal expansion: The representation of a number in base 10. A decimal point separates the places representing (to the left) 1s, 10s, 100s, and so on, and (to the right) the 1/10ths, 1/100ths, and so on.

dense: Rational numbers are dense within real numbers because between any two distinct real numbers, there is at least one rational number.

diagonalization: The method invented by Georg Cantor to prove that the cardinality of real numbers is greater than the cardinality of natural numbers.

distributive law: The arithmetic law that $a \times (b+c) = (a \times b) + (a \times c)$ for numbers a, b. and c.

e: The fundamental parameter in the measure of growth. The value of e is 2.71828... and is equal to the limiting value of the expression $\left(1+\frac{1}{n}\right)^n$ as n grows without bound.

Egyptian fraction: A fraction with numerator equal to 1.

element: A member of a collection.

empty set: The collection containing no elements.

equation: An expression that sets two quantities equal. For example, 2 + 2 = 4 and $x^2 - 2 = 0$ are equations.

Euler's formula: $e^{\pi i} + 1 = 0$

exponent: A superscript following a number or variable (e.g., the number 3 in the expression 2' = 2 x 2 x 2 is an exponent).

factor: A natural number m is a factor of a integer n if m divides evenly into n.

factorial: For a natural number a, a factorial is the product of the numbers from 1 up to and including, a. Denoted by n!.

Fibonacci numbers: The sequence of numbers 1, 1. 2. 3, 5. 8... in which each number alter the first two is equal to the sum of its two predecessors.

finite: A set is finite if the number of elements in the set is equal to a natural number.

fractal: An object that exhibits both infinite detail and self-similarity; as

portions are repeatedly magnified, more and more detail is revealed and patterns are repeated at different scales.

fundamental theorem of arithmetic: Every natural number greater than 1 can be written uniquely—up to reordering—as a product of prime numbers.

Gaussian integers: The collection of numbers of the form $a + bi$ where a and b can equal any integer and i is $\sqrt{-1}$

Gaussian prime: A Gaussian integer that cannot be written as the product of two smaller Gaussian integers.

Gelfond-Schneider Theorem: If an algebraic number not equal to 0 or 1 is raised to an algebraic irrational power, then the result is a transcendental number.

glyph: See pictograph.

Goldbach conjecture: Goldbach's conjecture states that every even number greater than 4 equals the sum of two primes.

golden ratio: The number $\frac{1+\sqrt{5}}{2}$.

Hilbert's problems: The list of open questions David Hilbert posed at the Congress of Mathematics in 1900. He considered them to be the most important open questions in mathematics for the 20th century.

Hindu-Arabic numerals: The numerals 0, 1. 2, 3, 4, 5, 6, 7. 8, and 9, developed from symbols used by Hindu mathematicians and brought to the West through Arab use.

i: $\sqrt{-1}$.

I Ching: An ancient Chinese system of philosophy and prediction based on a collection of symbols, called trigrams and hexagrams, that mimic a binary numeral system (though the symbols did not represent numbers).

imaginary numbers: The collection of numbers that are multiples of i. infinite: A set is infinite if it is not finite.

infinity: An abstract mathematical concept based on unbounded or unending quantities.

integers: The collection of numbers consisting of natural numbers (1, 2, 3 ...), together with all their negatives and 0.

irrational numbers: The collection of all numbers that are not rational. isosceles triangle: A triangle with at least two equal sides.

Jains (or Jana): A religious community in India dating back to 600 B.C.E. The Jains studied numbers extensively and even posited the existence of several sizes of infinity. Along with the Pythagoreans in Greece, they were one of the first groups to study numbers as abstract objects.

Lengua: An indigenous tribe from what is now Paraguay. The Lengua's number vocabulary included many words reflecting body parts.

limit of four: A conjecture that the human brain can deduce that a collection of items has four or fewer objects in it without actually counting the objects. For collections of five or more objects, most people have to truly count the items, however quickly, to determine how many there are.

logarithm: The exponent to which a base must be raised to produce a given number (e.g., the base-10 logarithm of 1000 is 3, because $10^3 = 1000$). When the base is e, the logarithm is called the natural logarithm.

magic square: A square array of the numbers 1. 2. 3...n2 in n rows and n columns so that the sums of each row, each column, and the two diagonals are all equal.

Maya: A civilization that thrived in Mesoamerica (present-day Central America and Mexico) from c. 1800 B.C.E. until 900 C.E., with continued presence until around 1600. The Mayans had a place-based numeral system in base 20 and were one of the earliest groups known to use a symbol for zero.

Mersenne prime: A prime number of the form $2^n - 1$.

Mesopotamia: The ancient region between the Tigris and Euphrates Rivers in what is now southern Iraq. Civilizations that flourished there are sometimes collectively called Mesopotamian.

multiplicative identity: The number 1 is the multiplicative identity because 1 x a = a for any number a.

multiplicative inverse: The multiplicative inverse of a nonzero number a is its reciprocals $\frac{1}{a}$, because $a \times \frac{1}{a} = 1$. the multiplicative identity (e.g., the multiplicative inverse of 5 is $\frac{1}{5}$, and the multiplicative inverse of $\frac{1}{2}$ is 2**).**

natural numbers: The collection of numbers 1, 2, 3, 4, 5 ...; also called the counting numbers.

nonrepeating expansion: A number expansion in any base is nonrepeating if it is not periodic.

normal number: A number is normal if it is normal in base 10 and that property holds analogously in expansions in all bases.

normal number in base 10: A number is normal in base 10 if its decimal expansion contains equal proportions of the digits 0, 1, 2 ... 9, as well as equal proportions of the two-digit expressions 00, 01, 02, 03 ... 99, as well as equal proportions of all three-digit, four-digit, five-digit expressions. and so on. for all finite-length expressions.

number: An ever-evolving mathematical concept involving quantity, measurement, and their abstractions and generalizations.

number line: A representation of the real numbers; a line extending endlessly in both directions, with a point marked as 0 and at least one more point. usually 1 . marking the unit of length. Each point on the line corresponds to a real number according to its distance from 0, with points to the right of 0 denoting positive numbers and points to the left of 0 denoting negative numbers.

number theory: The area of mathematics that focuses on the properties ar structure of numbers.

numeral system: A consistent system of symbols and notation for representing numerical values.

one-to-one correspondence: Two collections are said to be in a one-to-on correspondence if each item from one collection is paired with exactly one item from the other collection and vice versa.

ordinal number: A number that represents the position of an item in an ordered list (1^{st}, 2^{nd}, 3^{rd}...).

p-adic absolute value: The p-adic absolute value of a natural number in is where p is a specified prime number and n is the largest power of p that divides the number in.

p-adic numbers: The collection of all numbers in which all Cauchy sequences of all rational numbers approach a number in this collection under the p-adic absolute value.

pebble jar: A clay container made to hold number tokens or pebbles as a record of quantity.

perfect number: A number that is equal to the sum of its proper divisors; that is, the sum of all natural numbers less than that number that divide it evenly.

periodic expansion: A number expansion in any base is periodic if, eventually, the digits to the right of the decimal point fall into a pattern that

repeats forever. Also known as repeating expansion.

pi: The ratio of the circumference of a circle to its diameter. Pi is denoted by the Greek letter $\boldsymbol{\pi}$ and equals 3.14159....

pictograph: A character, drawing, or symbol used especially in early civilizations in Mesopotamia and Egypt. Also called a glyph.

place-based system: See **positional system**.

polynomial: An expression involving a single unknown (usually denoted by x) in which various powers of the unknown are multiplied by numbers, then added (e.g., $3x^2 - 17x + 5$ and $27x^8 + 7x3 - 8x$ are polynomials).

positional system: A numeral system in which the position of each symbol determines its value. Also known as a placed-based system. power set: The power set of a collection is the set containing exactly all subsets of the particular collection.

prime factorization: Calculation of all the prime factors in a number. prime number: A natural number greater than 1 that cannot be written as the product of two smaller natural numbers.

prime number theorem: The number of primes less than or equal to a particular natural number n is approximately $\frac{\ln(n)}{n}$, where ln(n) denotes the natural logarithm of n. As n increases without bound, the number of primes less than n gets arbitrarily close to $\frac{\ln(n)}{n}$

proof: A sequence of logical assertions, each following from the previous ones, that establishes the truth of a mathematical statement.

Pythagorean Theorem: $a^2+b^2=c^2$, given a right triangle with side lengths a. b, and c (with c the longest length—the hypotenuse).

Pythagoreans (Brotherhood): The community founded by Pythagoras in the 6^{th} century B.C.E. in what is now southern Italy. Members studied mathematics and philosophy, believing that numbers were fundamental to all reality. Along with the Jains in India, they were one of the first groups known to study numbers as abstract objects.

quadrivium: The four primary subjects studied by the Pythagoreans: arithmetic, geometry, music, and astronomy. These are considered by many to be the basis for the modern liberal arts.

radian measure: The measure of an angle equal to the length of the arc that the angle subtends on a circle of radius 1.

radix: The symbol, usually a period, that separates whole number digits from fractional digits in the decimal (or other base) expansion of a number.

ratio: A quantity that compares two measurements by dividing one into the other.

rational numbers: The collection of numbers consisting of all fractions (ratios) of integers with nonzero denominators.

real numbers: The collection of all decimal numbers, which together make up the real number line.

reed-stem stylus: A hollow reed with one end cut at an angle; used to mark wet clay with symbols for numerals and, later, cuneiform markings.

repeating expansion: See **periodic expansion**.

Rhind Papyrus: A papyrus scroll purchased by Scottish Egyptologist A. Henry Rhind in 1858. Dated to c. 1650 B.C.E., the scroll was copied by the scribe Ahmes from an original at least 200 years older. The Rhind Papyrus has been a critical document in understanding early Egyptian mathematics.

Riemann Hypothesis: A conjecture involving the complex number solutions to a particular equation. If true, the Riemann Hypothesis has important implications about the distribution of prime numbers. A prize of $1 million has been offered for a complete proof.

Roman numerals: A largely additive numeral system used in the Roman Empire employing capital letters as numeral symbols, including 1, V, X, L, D, C, and M.

sand table: A calculation device in which columns of pebbles or impressions were made in sand to do arithmetic; thought to be a precursor of the abacus.

set: A well-defined collection of objects whose members are called elements.

sine: The sine of an angle of a right triangle is the quotient of the length of the side opposite the angle divided by the length of the hypotenuse.

solution: Given an equation involving an unknown, a number is a solution to the equation if substituting that value for the unknown yields a valid equation.

square root: The square root of a number is a number that, when multiplied by itself, yields the first number.

square root of 2: $\sqrt{2}$, which equals 1.414....

subset: A collection is a subset of a set if every element in it is also an element in the set.

Sumerian: The earliest known civilization to inhabit the region of Mesopotamia in what is now southern Iraq. Dating primarily from about 5000 to 2000 B.C.E., the Sumerians invented a base-60 numeral system and cuneiform writing.

tally stick: A notched stick used for keeping records, especially in commerce, without using a specific numeral system.

ternary: See **base-3 numeral system**.

tetradys: The arrangement of 10 dots in a triangle, with rows of 1, 2, 3, and 4 dots. This figure and the number 10 had great significance to the Pythagoreans.

theorem: A mathematical statement dhat has been proven true using rigorous logical reasoning.

tokens: Molded clay or shaped stones used to represent quantities of particular items in days of early counting, beginning around 4000 B.C.E.

transcendental numbers: The collection of all numbers that are not algebraic.

Twin Prime Conjecture: The conjecture that states that there are infinitely many twin primes. Two prime numbers are twin primes if their difference is

uncountable: A set is uncountable if it is not countable.

unique factorization: Every natural number greater than 1 can be written as a product of prime numbers in only one way. up to a reordering of the ['actors. This product of primes is the unique factorization of the number.

Zeno's paradox: A scenario that suggests the impossibility of motion. When an arrow is shot toward a target, it must first reach the halfway point; before that, it must reach the point halfway to the halfway point; before that, it must reach the halfway point and so on. Because the arrow must travel infinitely many points before it even gets halfway to the target, it never gets there; thus, motion is impossible.

zero: The size of a collection having no members.

Biographical Notes

Ahmes (c. 1680 B.C.E.-c. 1620 B.C.E.). The Egyptian scribe who wrote the Rhind Papyrus—one of the oldest recovered mathematical documents. Though the text was authored 200 years earlier, Ahmes's copy has been critical in revealing work in early Egyptian mathematics and the important role of the scribe as a teacher and preserver of knowledge.

Archimedes (c. 287 B.C.E.-c. 212 B.C.E.). A Greek physicist and engineer, as well as a mathematician, Archimedes made many contributions to number theory and geometry. He calculated excellent approximations to pi given the arithmetic tools of the time and he proved that the area of a circle is r times the radius squared.

Bernoulli, Jacob (1654-1705). This Swiss mathematician was the first to approximate the value of e, having recognized this fundamental constant as a limiting value in a process of computing compound interest on an investment. One of eight mathematicians in his family, he made many contributions to the theory of probability.

Bhaskara I (600 C.E.-680 C.E.). This Indian mathematician is credited with the first use of what are now called the Hindu-Arabic numerals, including the symbol for zero.

Bhaskara II (1114-1185). Representing perhaps the height of mathematical knowledge of the $12^{1'}$ century, this Indian mathematician made many contributions to the study of equations and pondered infinite quantities. His work The Gem of Mathematics includes many story problems allegedly written to challenge and entertain his daughter.

Borel, Emil (1871-1956). This French mathematician was a pioneer in probability and a related area of analysis called measure theory. He introduced the concept of normal numbers as a measure of the randomness of the decimal expansion of real numbers.

Brahmagupta (598 C.E.-665 C.E.). The first known written record that acknowledges zero as a number was the work of this Indian mathematician and astronomer. He also understood many fundamental rules of arithmetic and the solving of equations.

Cantor, **Georg** (1845-1918). A German mathematician of Russian heritage and a student of Weierstrass, Cantor established much of the early fundamentals of set theory. Between 1874 and 1884, he created precise ways to compare in finite sets, establishing the existence of infinitely many sizes of infinity, as well as infinitely many irrational and transcendental numbers. The controversy stirred by his work, along with bouts of depression and

mental illness caused him great difficulties later in his life, and he died in a sanatorium.

Cardano, Girolamo (1501-1576). An Italian physician and mathematician, Cardano was the first to consider square roots of negative numbers, calling them "fictitious" numbers. He published solutions to the general cubic and quartic equations, acknowledging that the results themselves were credited to others.

Cauchy, Augustine-Louis (1789-1857). This French mathematician was particularly concerned with rigor and precision. He began the effort to prove the results of calculus rigorously and developed a definition of the real numbers using a limiting process involving rational numbers. The resulting infinite lists of numbers are called Cauchy sequences.

Cohen, Paul (1934-2007). An American mathematician at Stanford University who proved in 1963 that the Continuum Hypothesis cannot be proven true within the standard axioms of set theory. For this work, he won the Fields Medal in 1966, the closest award mathematics has to a Nobel Prize.

Dedekind, Richard (1831-1916). This German mathematician made significant contributions in abstract algebra and number theory and did fundamental work on the real numbers and infinite sets. He was an important friend and supporter of Georg Cantor during the time when Cantor struggled to have his work on infinite sets accepted.

Diophantus (c. 210 C.E.-c. 290 C.E.). This Greek mathematician lived in Alexandria. Egypt, where he wrote one of the earliest treatises on solving equations, Arithmetica. Though he considered negative numbers to be absurd and did not have a notation for zero, he was one of the first to consider fractions as numbers. In modern number theory, Diophantine analysis is the study of equations with integer coefficients for which integer solutions are sought.

Euclid (c. 325 B.C.E.-265 B.C.E.). A mathematician of Alexandria, Egypt. Euclid's major achievement was Elements, a set of 13 books on basic geometry and number theory. His work and style is still fundamental today, and his proofs of the infinitude of primes and the irrationality of ^{sqrt2}: are considered two of the most elegant arguments in all of mathematics.

Euler, Leonhard (1707-1783). A Swiss mathematician and scientist, Euler was one of the most prolific mathematicians of all time. He introduced standardized notation and contributed unique ideas to all areas of analysis, especially infinite sum formulas for sine, cosine, and e. The equation known as Euler's formula, e^{n}' + 1 = 0, is considered by many to be the most

beautiful in all mathematics

Fibonacci, Leonardo de Pisa (c. 1175-1250). An Italian mathematician, Fibonacci traveled extensively as a merchant in his early life. Perhaps the best mathematician of the 13th century, he introduced the Hindu-Arabic numeral system to Europe and discovered the special sequence of numbers that bears his name.

Gauss, Carl Friedrich (1777-1855). A German commonly considered the world's best mathematician. Gauss is known as the "Prince of Mathematics." He established mathematical rigor as the standard of proof and provided the first complete proof that complex numbers are algebraically closed, meaning that every polynomial equation with complex coefficients has its solutions among complex numbers.

Gelfond, Aleksandr (1906-1968). In 1934, this Russian mathematician answered the seventh of Hilbert's famous questions posed in 1900. The question was also answered independently in 1935 by Theodore Schneider: thus, the result is called the Gelfond-Schneider Theorem. Gelfond began teaching at Moscow State University in 1931, continuing there until the day he died. The Gelfond-Schneider Theorem can be used to show that e^{π}, known as Gelfond's constant, is transcendental.

Gödel, Kurt (1906-1978). Perhaps the most important of all logicians, Gödel was born and worked in Austria and Czechoslovakia but came to Princeton during early World War 11. His most famous work is known as Gödel's Incompleteness Theorem, which had profound implications for the logical foundations of mathematics. He also proved that the Continuum Hypothesis could not be disproved within the standard axioms of set theory.

Hermite, Charles (1822-1901). This French mathematician made important contributions to number theory and algebra. He spent his professional life at the Ecole Polytechnique and in the Faculty of Sciences of Paris and proved that e is transcendental in 1873. His technique was used by Lindemann to prove that π is transcendental.

Hilbert, David (1862-1943). Born in Prussia, this German mathematician was one of the most broadly accomplished and widely influential mathematicians in the late 19th and the 20'h century. He spent most of his professional life at the University of Göttingen, a top center for mathematical research. His presentation in 1900 of unsolved problems to the international Congress of Mathematics is considered to be one of the most important speeches ever given in mathematics. He was a vocal supporter of Georg Cantor's work and presented the Continuum Hypothesis as the first problem on his list in 1900.

Kronecker, Leopold (1823-1891). This German mathematician made contributions in number theory. algebra, and analytic ideas of continuity. As an analyst and logician, he believed that all arithmetic and analysis should be based on the integers and, thus, did not believe in the irrational numbers. This put him at odds with a number of colleagues and, especially, the new ideas of Cantor in the 1870s.

Lambert, Johann (1728-1777). The son of a poor tailor, Lambert was a German mathematician. astronomer, and physicist. In 1761, he gave the first proof that $\boldsymbol{\pi}$ is irrational. He studied geometry and the origins of the solar system spending the last 10 years of his life under the sponsorship of Frederick II of Prussia.

Leibniz, Gottfried (1646-1716). The son of a philosophy professor, Leibniz was highly influential as a mathematician, scientist, and philosopher. He discovered calculus independently of Newton and created the notation still in use today. He created the binary number system that is the basis of the computer and was the first to use the term transcendental.

Lindemann, Ferdinand von (1852-1939). A German mathematician and son of a language teacher, Lindemann is best known for his 1882 proof that $\boldsymbol{\pi}$ is transcendental. While a professor at the University of Konigsberg, he supervised the Ph.D. thesis of David Hilbert.

Lionville, Joseph (1809-1882). This French mathematician worked in many fields but is perhaps best known for his proof of the existence of transcendental numbers, given in 1844. He constructed actual examples and a special class of transcendental numbers is now called Liouville numbers.

Napier, John (1550-1617). A Scottish mathematician, physicist, and astronomer, Napier is best known for inventing the logarithm. He was the first to reference the number e and encouraged the use of the decimal point.

Oresme, Nicolas (1323-1382). Perhaps one of the most original thinkers of his time, Oresme was a French philosopher, mathematician, and scientist. Though he may have been the son of peasants, he became highly educated, ultimately serving as chaplain and advisor to the king of France. He was the first to consider numbers raised to fractional exponents and considered many other innovative ideas that presaged mathematical advances many centuries ahead of his time.

Pythagoras (c. 569 B.C.E.–c. 507 B.C.E.). Though best known for the theorem about right triangles that bears his name, Pythagoras had a much broader influence on mathematics and scholarship in general. Born on the Greek island of Samos, he moved to what is now southern Italy and

founded a religious and scholarly community called the Brotherhood. Because they left no written records, knowledge about these Pythagoreans comes from later sources, including Plato and Aristotle. The Brotherhood considered numbers the basis of all reality; Pythagoras is called the "Father of Number Theory." Together with the Jains in India, the Pythagoreans were the first to study numbers as abstract objects, opening the door to mathematics as an intellectual and creative pursuit.

Riemann, Bernhard (1826-1866). A major figure in mathematics during the mid-19th century, Riemann made important contributions to analysis, geometry, and topology. Calculus students everywhere know of the Riemann integral. His Ph.D. advisor was Gauss, and he spent his brief career at the University of Gottingen. His conjecture about the distribution of primes, called the Riemann Hypothesis, is one of the most important unsolved questions in mathematics today.

Schneider, Theodor (1911-1988). This German mathematician, who taught at the University of Gottingen, is best known for his 1935 solution to the seventh of Hilbert's questions posed in 1900. The question was also answered independently in 1934 by Aleksandr Gelfond; thus, the result is called the Gelfond-Schneider Theorem. The Gelfond-Schneider Theorem can be used to show that e^{π} is transcendental.

Stevin, Simon (1548/49-1620). A Flemish mathematician and engineer, Stevin wanted to bring about a second age of wisdom in which all earlier knowledge could be rediscovered. He discovered many fundamental results in physics and geometry and was a strong advocate for the adoption of the decimal system for numbers and coinage. He may have been the first to consider the number line.

Weierstrass, Karl (1815-1897). A German mathematician at the Technical University of Berlin, Weierstrass made many important contributions to calculus and analysis. Known as the "Father of Modem Analysis," he formalized the work of Bolzano and Cauchy to construct fundamental definitions still used in calculus today. He was a strong supporter of Georg Cantor.

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Answers to Selected Questions to Consider

Lecture Three, Question 1:

67, 34, 76, 20

Lecture Four, Question 2:

$2017 = (2 \times 10^3) + (0 \times 10^2) + (1 \times 10^1) + (7 \times 10°)$.
$1101, = (1 \times 2^3) + (1 \times 2^2) + (0 \times 21) + (1 \times 2°) = 8 + 4 + 1 = 13$ in base 10.
$212_3 = (2 \times 3^2) + (1 \times 31) + (2 \times 3°) = 18 + 3 + 2 = 23$ in base 10.

Lecture Five, Question 1:

The proper divisors of 28 are 1, 2, 4, 7, and 14; because
1 +2+4+7+- 14 = 28, 28 is perfect.

Observe that $1184 = 2\times 2\times 2\times 2\times 2\times 37$; thus, the proper divisors of 1184 are 1, 2, 4, 8, 16, 32, 37, 74, 148, 296, and 592;
$1+2+4+8+16+32+37+74+148+296+592 = 1210$.

Observe that $1210 = 2 \times 5 \times 11 \times 11$: thus, the proper divisors of 1210 are 1, 2, 5, 10. 11, 22, 55, 110, 121. 242, and 605:
$1+2+5+10+11+22+55+110+121+242+605 =1184$. Thus, 1184 and 1210 are amicable.

Lecture Five, Question 2:

1	14	8	11
15	4	10	5
12	7	13	2
6	9	3	16

Lecture Six, Question 1:

The table below shows that summing the squares of consecutive Fibonacci numbers yields the sequence 1 5, 13. 34. and so on. We recognize this as a list of every other Fibonacci number starting with 2. (This result can be proved in general.)

F_n	$(F_n)^2$	
1	1	
1	1	1+1 = 2
2	4	1 + 4 = 5
3	9	4 + 9 = 13
5	25	9 + 25 = 34
8	64	25 + 64 = 89
13	169	64+ 169 = 233
21	441	169 + 441 = 610
34	1156	441 + 1156 = 1597

Lecture Seven, Question 1:

Look at the number 2 x 3 x 5 x 7 x...x 1,000,000 + 1. Dividing this number by 2 or 3 or 5 and so on up to 1.000,000 will always give a remainder of 1; thus. no prime less than 1.000.000 divides this number. Although it is enormous, it is still a natural number: thus, it must either be prime or be factorable into primes, and therefore there must exist at least one prime number greater than 1,000.000.

Lecture Eight, Question 1:

We suppose $\sqrt{3}$ is rational and work toward a contradiction. If is

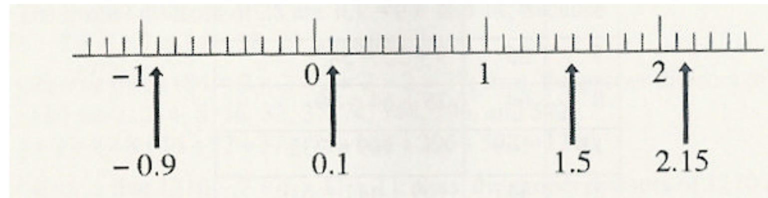
rational, then $\sqrt{3} = \frac{m}{n}$ for some integers in and n. Thus, 3 = $\frac{m^2}{n^2}$, and

$3n^2 = m^2$. Recall that every natural number can be written uniquely as a product of primes. Note also that 3 is prime and that 3 must appear an even number of times in the prime factorizations of m^2 and n^2; in that case, however, the equation $3n^2 = m^2$ would have an odd number of 3s dividing the left side and an even number of 3s dividing the right side, which is impossible. Thus, our original assumption must have been faulty, which means that $\sqrt{3}$ is irrational.

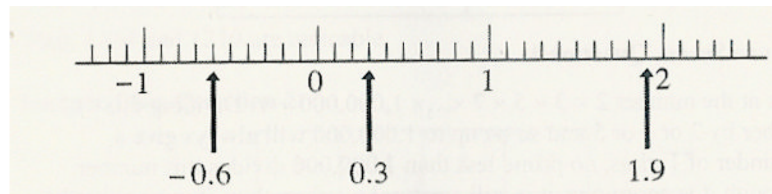
Lecture Eight, Bonus Challenge:

Unlike 2 and 3, 4 is not prime; rather, $4 = 2^2$. Thus, in the attempted proof, the equation $4n^2 = m^2$ becomes $2^2n^2 = m^2$. No contradiction is reached because there will be an even number of 2s dividing both sides of this equation.

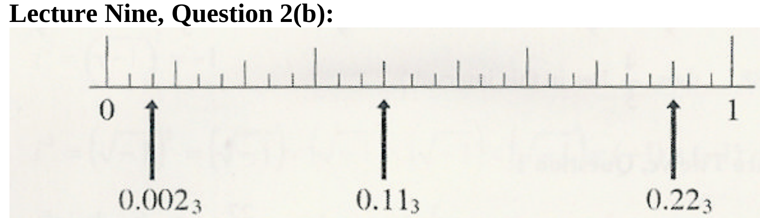
Lecture Nine, Question 1(a):



Lecture Nine, Question 1(b):



Lecture Nine, Question 2(b):



Lecture Ten, Question 1:

Only the second number, 3.7878787878..., is rational. It is rational because its decimal expansion repeats 78 forever and, thus, is periodic. The remaining three numbers are irrational. The number 0.10110111011110... will never be periodic because the number of 1s increases after each 0. The number 0.123456789101112... will never be periodic because its decimal digits are created by listing the natural numbers in increasing order, a list that never repeats. Finally, the number selected at random will never be periodic by the argument explained in the lecture.

Lecture Eleven, Question 1:

The numbers 0 and $\frac{1}{3}$ are in the Cantor set because they can be written in 3 base 3 using only 0s and 2s: $0 = 0.000..._3$ and $\frac{1}{3} = 0.1_3 = 0.222...._3$. The numbers $\frac{1}{2}$ and $\frac{4}{5}$ are not in the Cantor set. This is easiest to see using the geometric view of the Cantor set. The number $\frac{1}{2}$ lies in the middle third of the interval from 0 to 1 and, thus, is removed. The number $\frac{4}{5}$ lies in the middle third of the interval from $\frac{2}{3}$ to 1 and. thus, is also removed. (We see this last fact by noticing that the middle third of the interval from $\frac{2}{3}$ to 1 lies between $\frac{7}{9}$ and $\frac{8}{9}$ then observing that $\frac{7}{9} = 0.777$, $\frac{4}{5} = 0.8$, and $\frac{8}{9} = 0.888...$; thus. $\frac{4}{5}$ lies in this interval.)

Lecture Twelve, Question 1:

The expressions $5x^3 - 6x + 17$ and $\frac{3}{2}x^5 - 0.6x^2 + x - \frac{27}{13}$ are polynomials.

The other two expressions are not polynomials. One contains the variable x under a square root sign; the other involves the variable in a quotient.

Simplified numbers: $25^{3/2} = \left(\sqrt{25}\right)^3 = 5^3 = 125$, $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$.

Lecture Thirteen, Question 1:

First recall that 360° equals 2π radians: the circumference of a circle of radius 1. Then, because 45° is $\frac{1}{8}$ of 360°, it equals $\frac{1}{8}$ of 2π radians, or $\frac{\pi}{4}$ radians. Similarly, because 60° is $\frac{1}{6}$ of 360°, it equals $\frac{1}{6}$ of 2π radians. or $\frac{\pi}{3}$ radians. Finally, because 30° is $\frac{1}{12}$ of 360°, it equals $\frac{\pi}{6}$ radians.

Lecture Fifteen, Question 1:

The number $\sqrt[3]{2}$ is a solution to the equation $x^3 - 2 = 0$. When we substitute $\sqrt[3]{2}$ in place of x, we get $\left(\sqrt[3]{2}\right)^3 - 2 = 2 - 2 = 0$, which is a valid equation.

Lecture Sixteen, Question 2:

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

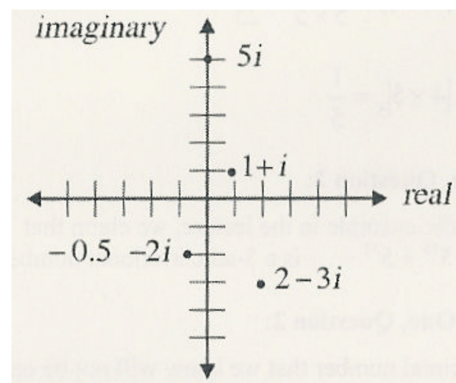
$$i^4 = \left(\sqrt{-1}\right)^4 = \left(\sqrt{-1}\right)\times\left(\sqrt{-1}\right)\times\left(\sqrt{-1}\right)\times\left(\sqrt{-1}\right) = (-1)\times(-1) = 1$$

$$i^5 = i^4 \times i = 1 \times i = i$$

$$i^{25} = i^{24} \times i = i^4 \times i^4 \times i^4 \times i^4 \times i^4 \times i^4 \times i = \left(i^4\right)^6 \times i = 1^6 \times i = 1 \times i = i$$

$$i^{1,000,000} = \left(i^4\right)^{250,000} = 1^{250,000} = 1$$

Lecture 17, Question 1:



Lecture Eighteen, Question 1:

$$|4|_2 = |2\times2| = \frac{1}{2\times2} = \frac{1}{4};$$

$$|12|_2 = |2\times2\times3| = \frac{1}{2\times2} = \frac{1}{4};$$

$$\left|\frac{1}{6}\right|_3=\frac{1}{|6|_3}=\frac{1}{|2\times3|_3}=\frac{1}{\frac{1}{2}}=2$$

$|0|_2=0$ by definition;

$$\left|\frac{24}{25}\right|_2=\frac{|24|_2}{|25|_2}=\frac{|2\times2\times2\times3|_2}{|5\times5|_2}=\frac{\frac{1}{2\times2\times2}}{1}=\frac{1}{8}$$

Lecture Nineteen, Question 1:

$$|5-0|_5=|5|=\frac{1}{5}$$

$$|\,25-0\,|_5=|\,25\,|_5=|\,5\times5\,|_5+\frac{1}{5\times5}=\frac{1}{25}$$

$$|\,25-5\,|_5=|\,20\,|_5=|\,4\times5\,|_5+\frac{1}{5}$$

Lecture Nineteen, Question 2:

Following the 3-adic example in the lecture, we claim that $5+5^2+5^4+5^8+5^{16}+5^{32}+\ldots$ is a 5-adic irrational number.

Lecture Twenty-One, Question 2:

To construct a decimal number that we know will not be on the list, we will choose digits to ensure that our new number differs from the first number on the list in its first digit, differs from the second number on the list in its second digit, differs from the third number on the list in its third digit, and so on. To simplify our choices, let us say that we will choose a 9 if the diagonal digit is not a 9 and a 0 if the diagonal digit is a 9; thus, the first seven decimal digits of our special number are: 0.9909090... .

By its construction, we know that this number differs from each number on the given list in at least one decimal place and. thus, cannot appear anywhere on the list.

Lecture Twenty-Two, Question 1:

There are 16 elements in the power set. In other words, the set {♣, ♠, ♥, ♦} has 16 subsets:

{},

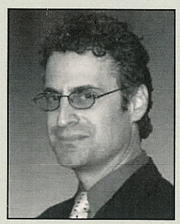
{♣}, {♠}, {♥}, {♦},

{♣, ♠}. {♣, ♥}, {♣, ♦}, {♠, ♥}, {♠, ♦}, {♥, ♦},

{♠, ♥, ♦}, {♣, ♠, ♦}, {♣, ♥, ♦}, (♠, ♥, ♦}.
{♣, ♠, ♥, ♦}

Lecture Twenty-Three, Question 1:

Suggest that she consider the power set of her collection. Cantor's Theorem guarantees that it will have a larger cardinality than her original set.



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